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INTEGER SEQUENCES WITH PROSCRIBED DIFFERENCES AND BOUNDED GROWTH RATE

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Let n, b and c be positive integers with $b \leq c$ and let $A = \{a_i\}_{i=0}^{n}$ be a sequence of integers such that $0 = a_0 < a_1 < \cdots < a_n$ and $a_{i+b} \leq a_i + c$ for all i with $0 \leq i \leq n-b$. We find all n, as a function of b, c and k, k a positive integer, so that all such sequences have no two members that differ by exactly k.

A chessmaster who has eleven weeks to prepare for a tournament decides to play at least one game every day, but in order not to tire himself he decides not to play more than twelve games during any one week. Then it is the case that for any k with $1 \leq k \leq 21$, there corresponds a succession of days during which the chessmaster will have played *exactly* k games.

If we let a_i be the number of games played during the first i days (and take $a_0 = 0$), then the above translates into a sequence problem that in one form or another has appeared in many places ([1, pp. 16 and 22], [2–6], [7, p. 74] [8–13]). In this paper we will completely solve the generalization introduced in [2] where 7 (days in a week) is replaced by b, 12 is replaced by c and 77 (total number of days of play) is replaced by n. We are thus involved with a sequence $A = \{a_i\}_{i=0}^{n}$ and the following conditions:

$$0 = a_0 < a_1 < a_2 < \cdots < a_n, \tag{1}$$

$$a_{i+b} \leq a_i + c \quad \text{for all } i,\ 0 \leq i \leq n-b, \tag{2}$$

$$a_j = a_i + k \quad \text{for some } i, j,\ 0 \leq i < j \leq n. \tag{3}$$

The problem. For what values of b, c, k and n do (1) and (2) imply (3)?

The original problem is easily solved by a simple application of the Pigeonhole Principle. By a slightly more sophisticated use of that principle [2], the problem was answered affirmately in a large number of cases; namely, if $n \geq \max(b, c-1, k)$. It follows that for the values $b = 7$, $c = 12$ and $n = 77$, (1) and (2) imply (3) if and only if $1 \leq k \leq 77$. The case $n = b$ was also fully solved in that

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paper, completing investigations by Hutchinson and Trow [7] (see also [13]) and Wang and Wu [12] that were motivated by an Olympiad problem (see [11]). Gilpin and Shelton [4] have since solved the $n=b$ case in an even more general setting where the single value of k was replaced by various finite sets.

Here, as in [2] and [4], we will switch the emphasis from k to n by seeking our answers in terms of n as a function of b, c and k. Toward this end we make the following definitions.

Definitions. Let b, c, k and n be positive integers and let $A=\{a_i\}_{i=0}^{n}$ be a sequence of nonnegative integers.

(a) A is called an n-sequence.

(b) A is called an (n, b, c)-sequence if (1) and (2) hold for A.

(c) A is called an $(n, b, c, \bar{k})$-sequence if it is an (n, b, c)-sequence for which (3) fails to hold.

(d) For $b \leqslant c$, the number $N(b, c, k)$ is defined by $N(b, c, k)=\min\{n \mid (3)$ holds for every (n, b, c)-sequence$\}$; if there is no such n we define $N(b, c, k)=\infty$.

We will frequently use the following properties of $N(b, c, k)$:

(a) If $n \geqslant N(b, c, k)$, then (3) holds for all (n, b, c)-sequences.

(b) If $0 \leqslant n < N(b, c, k)$, then $(n, b, c, \bar{k})$-sequences exist.

(c) $N(b, c, k) \geqslant b$ (otherwise (2) holds vacuously and so (3) is easily made to fail).

(d) $N(b, c, k) \geqslant k$ (otherwise (3) fails if we take $a_i=i$ for all i).

Examples given in [2] showed that $N=\infty$ is indeed possible and suggested that $c=2b$ is a pivotal case for that happenstance. Our first theorem confirms that impression by giving a finite upper bound (involving the g.c.d. of b and k; denoted by (b, k) here) for N in the cases not covered by those examples.

Theorem 1. *If $c>2b$ or if $c=2b$ and $k/(b, k)$ is odd, then $N(b, c, k)=\infty$; otherwise $N(b, c, k) \leqslant b+k-(b, k)$.*

Proof. To show that $N(b, c, k)=\infty$ we must produce, for each $n \geqslant \max(b, k)$, an $(n, b, c, \bar{k})$-sequence A.

For $c>2b$ and k even define A as follows:

$$a_i=\begin{cases} 2i & \text{for } 0 \leqslant i < k/2, \\ 2i+1 & \text{for } k/2 \leqslant i < k, \\ a_{i-k}+2k & \text{for all } i \text{ with } k \leqslant i \leqslant n. \end{cases}$$

For $c \geqslant 2b$ and k odd, indeed for k/d odd where d is any divisor of (k, b), define A by taking

$$a_{md+r}=2md+r \quad \text{for } 0 \leqslant r < d,\ 0 \leqslant md+r \leqslant n.$$

It is shown in [2], and is a nice number-theoretic exercise, that these are $(n, b, c, \bar{k})$-sequences, thus completing the proof of the first part of the theorem.

We note that the second of these examples has the same periodic structure displayed by the first; namely, that $a_{i+k} = a_i + 2k$ for all i, $0 \leq i \leq n-k$. We will soon see that these examples are not exceptional in this regard. But more immediately, in order to complete the proof of Theorem 1, we need a related result that is not only useful in constructing $(n, b, c, \bar{k})$-sequences but in giving proofs (by the method of contradiction) that (1) and (2) imply (3).

Lemma 1 ([2]). *If A is an $(n, b, c, \bar{k})$-sequence, then $a_{i+k} \geq a_i + 2k$ for all i with $0 \leq i \leq n-k$.*

Proof. For each i, the numbers $a_{i+1}, a_{i+2}, \ldots, a_{i+k}$ include at most one each from each of the sets $\{a_i+1, a_i+k+1\}$, $\{a_i+2, a_i+k+2\}, \ldots, \{a_i+k-1, a_i+2k-1\}$ and neither a_i nor a_i+k. The lemma follows by the Pigeonhole Principle. (Note that the proof did not actually use condition (2).) □

Because they are so frequently used we list together here (for easy reference) three important inequalities. The first is equivalent to (1) and the second is just (2) expressed in a manner uniform with the other two.

Basic Inequalities. If A is an $(n, b, c, \bar{k})$-sequence, then

$$a_i \leq a_{i+m} - m \quad \text{for } 0 \leq i < i+m \leq n \quad \text{and} \quad a_0 = 0, \tag{1'}$$

$$a_i \leq a_{i-b} + c \quad \text{for } b \leq i \leq n, \tag{2'}$$

$$a_i \leq a_{i+k} - 2k \quad \text{for } 0 \leq i \leq n-k. \tag{3'}$$

We now return to the remaining cases in Theorem 1. Let A be an $(n, b, c, \bar{k})$-sequence and suppose that $0 = s_0, s_1, \ldots, s_m$ is a sequence of subscripts of A (so all are within the interval $[0, n]$) such that each s_i is obtained from s_{i-1} by either adding b or subtracting k. Thus $s_1 = b$ and $s_m = pb - qk$ for $p > 0$, $q \geq 0$. We first consider the situation when $c < 2b$. If $s_{m-1} = (p-1)b - qk$, then $a_{pb-qk} \leq a_{(p-1)b-qk} + c < a_{(p-1)b-qk} + 2b$, while $s_{m-1} = pb - (q-1)k$ implies that $a_{pb-qk} \leq a_{pb-(q-1)k} - 2k$. In either case we conclude, by induction on m, that $a_{pb-qk} < 2(pb-qk)$. Similarly, we conclude that if we get to $rk - sb$ with $r > 0$ and $s \geq 0$ by a sequence of integers all in $[0, n]$ and with each obtained from the last by either adding k or subtracting b, then $a_{rk-sb} \geq 2(rk-sb)$.

But it is a simple number-theoretic fact that we can get to (b, k) by each of the above methods if $n \geq b + k - (b, k)$. It follows that, if $c < 2b$, then $N(b, c, k) \leq b + k - (b, k)$.

If $c = 2b$ and $k/(b, k)$ is even, then $k/2$ is a multiple of (b, k) and since $k/2 < k \leq k + b - (b, k)$ we can also get to $k/2$ by each of the above methods. However, since we do not have $c < 2b$, but $c = 2b$ we can only conclude that

$k=2(k/2)\leqslant a_{k/2}\leqslant 2(k/2)=k$ by the above inequalities. However, this means that $a_{k/2}=k$, which means (3) holds, so we conclude that $N(b, 2b, k)\leqslant b+k-(b, k)$ then as well. That completes the proof of Theorem 1. $\square$

We now take up an important reduction technique. Its proof will be based on the following three lemmas.

Lemma 2. *If* $n=N(b, c, k)-1<\infty$, *then there is an* $(n, b, c, \bar{k})$*-sequence* A *such that*

(a) $a_{n-k+1}>a_n-2k$ *and*

(b) *for* $a_n-k<x\leqslant a_n$ *we have* $x\in A$ *if and only if* $x-k\notin A$.

Proof. By the choice of n, there is an $(n, b, c, \bar{k})$-sequence A. Suppose that $a_{n-k+1}\leqslant a_n-2k$. Then we will modify A as follows: of the $2k$ numbers in

$$S=\{a_n-2k+1, a_n-2k+2, \ldots, a_n\}$$

over half are not members of A since $a_{n-k+2}, \ldots, a_n$ are the only members of A possibly in that interval. But then, since (3) does not hold for A, two of these nonmembers of A differ by k, say a_n-j and a_n-j-k with $1\leqslant j\leqslant k-1$. Now replace a_n by a_n-j and resubscript. Obviously (1) and (2) hold for the new sequence and (3) fails to holds since $(a_n-j)+k\geqslant a_n-(k-1)+k=a_n+1$ and of course $(a_n-j)-k\notin A$ by our choice of j.

Repetition of this modification procedure must eventually produce an $(n, b, c, \bar{k})$-sequence A with $a_{n-k+1}>a_n-2k$. But then, by Lemma 1, we have $a_{n-k}\leqslant a_n-2k<a_{n-k+1}$ and so exactly half of the numbers from S are in A. Property (b) now follows since (3) fails to hold for A. $\square$

Lemma 3 (The Extension Lemma). *If* $c\leqslant 2b$, $n=N(b, c, k)-1<\infty$ *and* a *is an* $(n, b, c, \bar{k})$*-sequence with* $a_{n-k+1}>a_n-2k$, *then the sequence* $A^{\mathrm{Ext}}=\{a_i\}_{i=0}^{n+k}$ *with* $a_j=a_{j-k}+2k$ *for* $n+1\leqslant j\leqslant n+k$, *is an* $(n+k, b+k, c+2k, \bar{k})$*-sequence.*

Proof. Since $a_{n-k+1}+2k>a_n$, (1) clearly holds for A^{Ext} and the only way (3) could hold is to have $a_i+k=a_j$ for some i and j with $i\leqslant n<j$. But then we have $0\leqslant j-k\leqslant n$ since $k\leqslant n<j\leqslant n+k$, and so $a_i=a_j-k=(a_{j-k}+2k)-k=a_{j-k}+k$ which is a contradiction.

So consider (2) w.r.t. $b'=b+k$ and $c'=c+2k$. First note that $n<b'$ since, by Theorem 1, we have $n+k=N(b, c, k)-1+k\leqslant(b+k-(b, k))-1+k=b'+k-(b, k)-1<b'+k$. Thus, for $b'+i\leqslant n+k$ we have $b+i\leqslant n$ and so $a_{b'+i}=2k+a_{b'+i-k}=2k+a_{b+i}\leqslant 2k+a_i+c=a_i+c'$.

Thus A^{Ext} is an $(n+k, b+k, c+2k, \bar{k})$-sequence as was to be proved. Since we will want to apply the Extension Lemma to A^{Ext} we further note that with $n'=n+k$ we have $a_{n'-k+1}=a_{n-k+1}+2k>(a_n-2k)+2k=a_n=a_{n+k}-2k=a_{n'}-2k$. And of course the complementary nature of the end of A is not only preserved but extended; that is we now have for $a_n - k < x \leqslant a_{n+k}$, $x \in A^{\text{Ext}}$ if and only if $x - k \notin A^{\text{Ext}}$. □

Lemma 4 (The Truncation Lemma). *If $n = N(b, c, k) - 1 < \infty$ with $k < b$ and $2k < c$ and if A is an $(n, b, c, \bar{k})$-sequence, then the sequence $A^{\text{Trun}} = \{a_i'\}_{i=0}^{n-k}$, with $a_i' = a_{i+k} - a_k$ is an $(n-k, b-k, c-2k, \bar{k})$-sequence.*

Proof. Obviously (1) holds and (3) fails for A^{Trun} so consider a_i' and $a_{i+b''}'$ with $0 \leqslant i < i + b'' \leqslant n - k$ where $b'' = b - k$ and $c'' = c - 2k$. Then $a_{i+b''}' = a_{i+(b-k)+k} - a_k = a_{i+b} - a_k \leqslant a_i + c - a_k \leqslant a_{i+k} - 2k + c - a_k = a_i' + c''$. □

Theorem 2 (The Reduction Theorem). *If $2k < c \leqslant 2b$, then*

$$N(b, c, k) = k + N(b-k, c-2k, k).$$

Proof. We note that the claim is correct if $c = 2b$ and $N(b, c, k) = \infty$. So assume that $N(b, c, k) < \infty$.

Now let $n = N(b, c, k) - 1$ and let A be an $(n, b, c, \bar{k})$-sequence. Then A^{Trun} is an $(n-k, b-k, c-2k, \bar{k})$-sequence. Thus $k + N(b-k, c-2k, k) \geqslant k + (n - k + 1) = N(b, c, k)$.

On the other hand, let $n = N(b-k, c-2k, k) - 1$ (this is finite by Theorem 1) and let A be an $(n, b-k, c-2k, \bar{k})$-sequence. Then by Lemmas 2 and 3 there are $(n+k, b, c, \bar{k})$-sequences. Thus $N(b, c, k) \geqslant n + k + 1 = k + N(b-k, c-2k, k)$. □

In the following it is helpful to observe that $k\lfloor b/k \rfloor$ is the symbolic form of the largest multiple of k in b.

Corollary. *If $c \leqslant 2b$, $k\lfloor b/k \rfloor \leqslant c - b$ and $k \nmid b$, then*

$$N(b, c, k) = k\left\lfloor \frac{b}{k} \right\rfloor + N\left(b - k\left\lfloor \frac{b}{k} \right\rfloor, c - 2k\left\lfloor \frac{b}{k} \right\rfloor, k\right).$$

Proof. If we let $c_i = c - 2ki$ and $b_i = b - ki$, then $c_i \leqslant 2b_i$, so we can repeatedly apply the Reduction Theorem as long as we have $c_i > 2k$. But, from our hypothesis, we have $c - 2k\lfloor b/k \rfloor \geqslant b - k\lfloor b/k \rfloor \geqslant 1$, so the claim follows. □

We observe that if $k \mid b$, then $b - k\lfloor b/k \rfloor = 0$; but we still get that $N(b, c, k) = k(\lfloor b/k \rfloor - 1) - N(k, c', k)$ for some $c' \geqslant k$. Soon we will see that $N(k, c', k) = k$; hence, if $k \mid b$, then $N(b, c, k) = k\lfloor b/k \rfloor = b$.

The Reduction Theorem also enables us to construct, for $n = N(b, c, k) - 1 < \infty$ and $c \leqslant 2b$, especially 'nice' $(n, b, c, \bar{k})$-sequences. For let A be a sequence as in Lemma 2. Extend it repeatedly by Lemma 3 until the extension portion is as long as the original sequence, then apply Lemma 4 to that sequence the same number of times. By the Reduction Theorem the result is an $(n, b, c, \bar{k})$-sequence. It has

Table 1

$N=26$, $b=22$, $c=35$, $k=24$:
0 1 13 14 15 16 17 18 19 20 21 22 23 26 27 28 29 30 31 32 33 34 35 36 48 49
$N=50$, $b=46$, $c=83$, $k=24$:
0 1 13 14 15 16 17 18 19 20 21 22 23 26 27 28 29 30 31 32 33 34 35 36 48 49 61 62 63 64 65 66 67 68 69 70 71 74 75 76 77 78 79 80 81 82 83 84 96 97

the properties that

(a) $a_{i+k}=a_i+2k$ for all i, $0 \leq i \leq n-k$, and

(b) for $0 \leq x \leq a_n - k$, $x \in A$ if and only if $x+k \notin A$.

Note that property (b) is in fact redundant; we have included it here to emphasize the complementary nature of these examples as well as their periodic form. Two such sequences are given in Table 1. The repetitive and complementary nature of these examples is more obvious if we note that they are respectively of the form

$$\underline{2},\ 11,\ \underline{11},\ 2,\ \underline{11},\ 11,\ \underline{2}$$

and

$$\underline{2},\ 11,\ \underline{11},\ 2,\ \underline{11},\ 11,\ \underline{2},\ 11,\ \underline{11},\ 2,\ \underline{11},\ 11,\ \underline{2}$$

where $\underline{a}, b, \underline{c}, \ldots$ means the sequence uses the first a integers (starting at 0), misses the next b integers, uses the next c integers, etc.

The following lemma gives another reduction technique whose proof is omitted as it is straightforward.

Lemma 5 (The Multiplication Inequality). *If $m \geq 1$ and $N(b,c,k)<\infty$, then*

$$N(mb, mc, mk) \geq mN(b,c,k).$$

We are now positioned to finish the $c=2b$ case.

Theorem 3 ($c=2b$). *We have*

$$N(b, 2b, k)=\begin{cases}\infty & \text{if } k/(b,k) \text{ is odd,}\\ b+k-(b,k) & \text{if } k/(b,k) \text{ is even.}\end{cases}$$

Proof. From Theorem 1, the corollary to the Reduction Theorem and the Multiplication Inequality we see that it suffices to prove that $N(b,2b,k) \geq b+k-(b,k)$ when $(b,k)=1$, k is even and $b<k$.

The proof will be completed by constructing an $(n, b, 2b, \bar{k})$-sequence with $n=b+k-2$. Toward that end we define a digraph D by taking $V(D)=\{0,1,\ldots,k-2\}$ and $E(D)=\{(i,j) \mid i,j \in V(D), j-i \in \{b, -(k-b)\}\}$.

It is easily seen that every vertex has indegree and outdegree equal to 1, except that $b-1$ has indegree 0 and $k-b-1$ has outdegree 0. If $b-1=k-b-1$ we have $k = 2b$ and hence $b = 1$ and $k = 2$ since $(k, b) = 1$. The theorem is correct in this case so we assume that $k > 2$. Thus D consists of one directed path (from $b-1$ to $k-b-1$) and zero or more directed cycles. Suppose that $v_0, v_1, \ldots, v_m = v_0$ is such a cycle with $m > 0$. By the definition of D, $v_m = v_0 + n_1 b - n_2(k-b)$, where $n_1, n_2 \geq 0$ and $n_1 + n_2 = m$. But $v_m = v_0$ so $n_1, n_2 > 0$ and $n_1 b - n_2(k-b) = 0$, which implies that $b \mid n_2$ and $(k-b) \mid n_1$ since $(b, k-b) = (b, k) = 1$. Therefore $n_1 + n_2 \geq (k-b) + b = k$, contradicting the fact that $|V(D)| = k-1$. Hence no such cycles exist, and D consists of a single directed path $t_0, t_1, \ldots, t_{k-2}$, where $t_0 = b-1$ and $t_{k-2} = k-b-1$.

Another important observation about D is that $t_j + t_{k-2-j} = k-2$ for $0 \leq j \leq k-2$. This is easily proved by induction on j. (For example, $t_1 = t_0 + b$ and $t_{k-2} = t_{k-3} - (k-b)$ imply that $t_1 + t_{k-3} = 2(k-1)$ which is a contradiction.) In particular, $t_{k/2-1} = k/2 - 1$. Now define $A = \{a_i\}_{i=0}^{n}$ thus:

$$a_i = \begin{cases} 2i+1 & \text{if } i = t_j \text{ with } 0 \leq j \leq k/2 - 1, \\ 2i & \text{if } i = t_j \text{ with } k/2 \leq j \leq k-2, \\ 2k-2 & \text{if } i = k-1, \\ a_{i-k} + 2k & \text{if } k \leq i \leq n. \end{cases}$$

While it is nontrivial to show that $\{a_i - a_0\}_{i=0}^{n}$ is an $(n, b, c, \bar{k})$-sequence, that demonstration is not particularly enlightening so we omit it. □

Hence we are left with the cases where $c < 2b$; these fall into two subcases depending on which of $k\lfloor b/k \rfloor$ and $c - b$ is greater.

Theorem 4. *For $k \leq b \leq c < 2b$, we have $N(b, c, k) = b$ if and only if $k\lfloor b/k \rfloor > c - b$.*

Proof. Let $m = \lfloor b/k \rfloor$; hence $b = mk + r$, $0 \leq r < k$.

First assume that $mk > c - b$ and suppose that A is an $(n, b, c, \bar{k})$-sequence with $n \geq b$. Thus $a_{mk} = a_{b-r} \leq a_b - r \leq c - r = c + mk - b < 2mk$ which contradicts Lemma 1. We conclude that $N(b, c, k) \leq b$ and hence that $N(b, c, k) = b$.

Conversely, assume that $mk \leq c - b$. Then define the b-sequence A by taking $a_{qk+p} = 2qk + p$ for $0 \leq p < k$ and $0 \leq qk + p \leq b$. Thus $a_b = 2mk + r = mk + b \leq c$, so (2) holds for A. And clearly (1) holds and (3) does not so A is a $(b, b, c, \bar{k})$-sequence. Hence $N(b, c, k) > b$ in this situation. This completes the proof of Theorem 4. □

However, for $k \leq b \leq c < 2b$ Theorem 4 leaves unsolved the case with $k\lfloor b/k \rfloor \leq c - b$. But then $k \nmid b$, so by the corollary to the Reduction Theorem, we get $N(b, c, k) = k\lfloor b/k \rfloor + N(b', c', k)$ where $b' = b - k\lfloor b/k \rfloor < k$. The solution of our problem is thus finished by the following theorem.

Theorem 5. *If* $b \leqslant c < 2b$, $b < k$ *and*

$$B(b, c, k) = \min\left\{n \mid n + (b + k - c)\left\lfloor\frac{n-k}{k-b}\right\rfloor \geqslant 2(c-b)\right\},$$

then

$$N(b, c, k) = B(b, c, k)$$
$$= \begin{cases} 2(c-b) - (b+k-c)\left\lfloor\frac{c-b}{2k-c}\right\rfloor & \text{if } \left\lfloor\frac{c-b}{2k-c}\right\rfloor < \left\lfloor\frac{c-2b+k-1}{2k-c}\right\rfloor, \\ k + (k-b)\left\lfloor\frac{c-b}{2k-c}\right\rfloor & \text{otherwise.} \end{cases}$$

Proof. We will first establish that $N(b, c, k) = B(b, c, k)$.

Let A be an $(n, b, c, \bar{k})$-sequence with $n \geqslant k > b$ and let r be the number with $a_r < k < a_{r+1}$, which exists since $a_0 = 0 < k < a_k$.

Claim. $n \leqslant r + c - b$.

Proof. Suppose to the contrary that $n \geqslant r + c - b + 1$. We will also need that $r + c - 2b + 1 \geqslant 0$ so that we can use it as a subscript in dealing with A. But, since $0 < 2b - c = b - (c-b) \leqslant b \leqslant n$, we have $a_{2b-c} \leqslant a_b - (c-b) \leqslant c - (c-b) = b \leqslant k$, and so $2b - c \leqslant r$. Using the Basic Inequalities and the definition of r we then have

$$a_{r+c-b+1} \leqslant a_{r+c-2b+1} + c \leqslant a_r + (c - 2b + 1) + c < k + 2(c-b) + 1,$$

and

$$a_{c-b+1} + k \leqslant a_b - (2b - c - 1) + k \leqslant c - 2b + c + 1 + k = 2(c-b) + k + 1.$$

Thus the $2(c - b + 1)$ numbers

$$a_1 + k, a_2 + k, \ldots, a_{c-b+1} + k, a_{r+1}, a_{r+2}, \ldots, a_{r+c-b+1}$$

all lie in the set $\{k+1, \ldots, 2(c-b)+k+1\}$ which only has $2(c-b+1)-1$ elements. This contradicts the assumption that A has no difference of k and thus establishes the claim. $\square$

Now $a_k \geqslant 2k$ by Lemma 1, so $a_{k-b} \geqslant 2k - c$ which implies that $a_{2k-b} \geqslant 4k - c$ and so $a_{2(k-b)} \geqslant 2(2k-c)$. In general, we have $a_{s(k-b)} \geqslant s(2k-c)$ if $0 \leqslant s(k-b) \leqslant n - b$. Thus, taking the maximum value of s possible, we have $a_j \geqslant (2k-c)\lfloor(n-b)/(k-b)\rfloor$ where $j = (k-b)\lfloor(n-b)/(k-b)\rfloor \leqslant n - b$, which by the Claim, gives $j \leqslant r + c - 2b < r$. But $r - j \leqslant a_r - a_j$ so $r \leqslant j + a_r - a_j \leqslant (k-b)\lfloor(n-b)/(k-b)\rfloor + (k-1) - (2k-c)\lfloor(n-b)/(k-b)\rfloor = k - 1 - (b+k-c) \times \lfloor(n-b)/(k-b)\rfloor$. Using the Claim again and the fact that $\lfloor(n-b)/(k-b)\rfloor = \lfloor(n-k)/(k-b)\rfloor + 1$ we have $n + (b+k-c)\lfloor(n-k)/(k-b)\rfloor \leqslant 2(c-b) - 1$. Thus $N(b, c, k) \leqslant B(b, c, k)$.

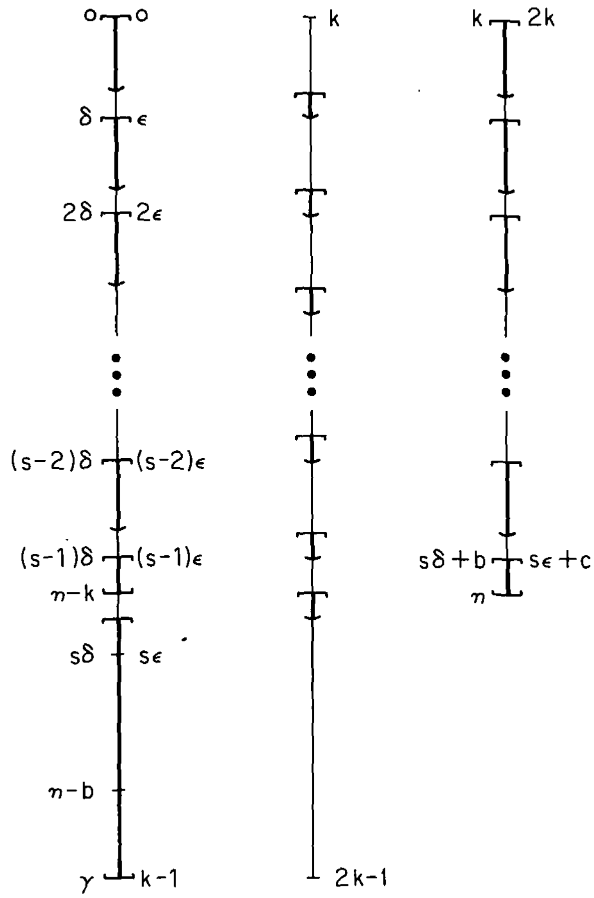

Fig. 1.

To show that $N(b, c, k) \geqslant B(b, c, k)$ we construct an $(n, b, c, \bar{k})$-sequence with $n = B(b, c, k) - 1$. Not surprisingly, it will have the periodic and complementary structure described in the paragraph following the corollary to the Reduction Theorem. Its description may be more easily comprehended by aid of Fig. 1 where $\delta = k - b$, $\varepsilon = 2k - c$ and along the line, sequence values are on the right, subscripts on the left. The line is drawn in three segments in order to better show the periodic and complementary nature of these examples.

By our choice of n we have $n + (b + k - c)\lfloor (n+k)/(k-b) \rfloor \leqslant 2(c - b) - 1$. Setting $s = \lfloor (n-b)/(k-b) \rfloor$ and substituting $\lfloor (n-k)/(k-b) \rfloor = s - 1$ and $b + k - c = (2k - c) - (k - b)$ into this inequality we get $s(2k - c) \leqslant (k-1) + s(k-b) - (n-b) + (c - 2b)$, or since $c < 2b$, $s(2k-c) + ((n-b) - s(k-b)) < k - 1$. Since the definition of A to follow will have $a_{s(k-b)+j} = s(2k-c) + j$ for $0 \leqslant j \leqslant (k-1) - s(2k-c)$, this means that $a_{n-b} < k - 1$. Furthermore, since

$(n-b)-(n-k)=k-b$ it also means, by the definition of s, that $(s-1)(k-b)\leqslant n-k<s(k-b)$. This location of $n-k$ is critical to our definition of A; in fact we will define a_1 through a_r so that $a_r=k-1$ and then extend that on through to a_n by properties (a) and (b) given after the corollary to the Reduction Theorem.

So we take $a_{m(k-b)+i}=m(2k-c)+i$ for $0\leqslant i<k-b$ and $0\leqslant m\leqslant s-2$, $a_{(s-1)(k-b)+i}=(s-1)(2k-c)+i$ for $0\leqslant i$ until we get to a_{n-k} and then take $a_{n-k+1}, a_{n-k+2},\ldots,a_{s(k-b)},\ldots,a_{n-b},\ldots$ up to $k-1$ as the string of consecutive integers having $a_{s(k-b)}=s(2k-c)$. Thus, by our previous definition of r, excluding the case $s=0$, $a_r=k-1$. Now for $k\leqslant x\leqslant 2k-1$, we take $x\in A$ if and only if $x-k\notin A$, thus defining A up to a_{k-1}. Finally, for $k\leqslant i\leqslant n$, we define $a_i=a_{i-k}+2k$. It is important to note that this last extension only uses the portion of A up to a_{n-k} (see Fig. 1).

Clearly (1) holds for A (as $2k-c>k-b$, since $k>b$ and $c<2b$), while (3) fails to hold; we need only to verify (2). We begin by noting that

$$\begin{aligned}a_{n-k+1}&=|\{x\mid x\in A,\ 1\leqslant x\leqslant a_{n-k+1}\}|+|\{x\mid x\notin A,\ 1\leqslant x\leqslant a_{n-k+1}\}|\\&=(n-k+1)+(s(2k-c)-s(k-b))\\&=n+s(b+k-c)-k+1.\end{aligned}$$

If $s=0$, then $n<k$. The sequence A now has the degenerate form $\{i\}_{i=0}^{n}$ which clearly satisfies (2).

If $s=1$, then $\lfloor(n-k)/(k-b)\rfloor=0$ and so, by the definition of $B(b,c,k)$, $n\leqslant 2(c-b)-1$. Thus A has four blocks of consecutive values as follows (where a_{n-k+1} is found from the above formula): $a_0=0,\ldots,a_{n-k}=n-k$; $a_{n-k+1}=n+b-c+1,\ldots,a_{c-b-1}=k-1$; $a_{c-b}=n+1,\ldots,a_{k-1}=n+b+k-c$; $a_k=2k,\ldots,a_n=n+k$. As we saw before, a_{n-b} lies in the second block, so the only potential threat to (2) is the difference between a_{n-k} and a_{n-k+b}. The latter element is easily seen to lie in the third block, so $a_{n-k+b}=a_{c-b}+(n-k+b)-(c-b)=2n+1-k+2b-c$. Therefore, $a_{n-k+b}-a_{n-k}=n+1+2b-c$, which is at most c because $n\leqslant 2(c-b)-1$.

Finally, suppose $s\geqslant 2$. For $1\leqslant m\leqslant s-2$ and $0\leqslant j\leqslant k-b-1$ we have $a_{m(k-b)+b+j}=a_{(m-1)(k-b)+k+j}=2k+a_{(m-1)(k-b)+j}=2k+(m-1)(2k-c)+j=m(2k-c)+c+j=a_{m(k-b)+j}+c$. Similarly, $a_{i+b}\leqslant a_i+c$ for $(s-1)(k-b)\leqslant i\leqslant r$. The only possible problem is with $a_{i+b}-a_i$ for $0\leqslant i\leqslant k-b-1$ and clearly this will be at most c if $a_{k-1}-a_{k-b-1}\leqslant c$. But $a_{k-b-1}=k-b-1$ and $a_{k-1}=k+(a_{n-k+1}-1)=n+s(b+k-c)$, so (2) holds if $n+(s-1)(b+k-c)\leqslant 2(c-b)-1$. But that is precisely what our choice of $n=B(b,c,k)-1$ gave us.

It remains to solve the inequality in the definition of $B(b,c,k)$. At the suggestion of the referee (and we thank him/her for several useful suggestions) that is left as an exercise for the reader. This, and other omitted portions, can be obtained from either author. □

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References

[1] R.A. Brualdi, Introductory Combinatorics (North-Holland, New York, 1977).
[2] P. Erdös, R. L. Hemminger, D.A. Holton and B.D. McKay, On the Chessmaster Problem, Proc. Silver Jubilee Conference, University of Waterloo, June 1982.
[3] M. Gilpin and R. Shelton, Problem 1152, Math. Mag. 55 (1982) 237.
[4] M. Gilpin and R. Shelton, Forced differences between terms of subsequences of integer sequences, preprint.
[5] M. Gilpin and R. Shelton, Predicates with periodic maximal length functions, preprint.
[6] R.L. Hemminger and B.D. McKay, More on the Chessmaster Problem, Proc. 14th Southeastern Conference on Combinatorics, Graph Theory and Computing, Florida Atlantic University, February 1983.
[7] J.P. Hutchinson and P.B. Trow, Some Pigeonhole Principle results extended, Amer. Math. Monthly 87 (1980) 648–651.
[8] C.L. Liu, Topics of Combinatorial Mathematics (MAA, Washington, DC, 1972).
[9] C.L. Liu, Elements of Discrete Mathematics (McGraw-Hill, New York, 1977).
[10] K. R. Rebman, The Pigeonhole Principle, Two-year College Math. J. 19 (1979) 3–13.
[11] D.O. Shklarsky, N.N. Chentzov and I.M. Yaglom, The USSR Olympiad Problem Book (Freeman, San Francisco, 1962).
[12] Y.F. Wang and L.G. Wu, A note on the Pigeonhole Principle (Chinese: English summary), J. Math. Res. Exposition 1 (1981), 103–106.
[13] W.D. Wei, Review #82g:0516, Math. Rev. (July, 1982).