On an Edge Crossing Problem

Peter Eades,

Department of Computer Science
University of Queensland
St. Lucia, Queensland 4067

Brendan D. McKay,

Department of Computer Science
Australian National University
G.P.O. Box 4, Canberra, A.C.T. 2601

Nicholas C. Wormald

Department of Mathematics
University of Auckland
Private Bag, Auckland
NEW ZEALAND

ABSTRACT

A ***bipartitioned graph*** $G = (L, R, E)$ consists of two sets L (the "left side") and R (the "right side") of vertices and a set $E \subseteq L \times R$ of edges. Such graphs are commonly drawn with L on a vertical line on the left side of the page, and R on a vertical line on the right. These drawings are more useful if they have few edge crossings. In this paper we give a fast algorithm for drawing bipartitioned graphs without edge crossings, if possible. Further, we show that the problem of minimizing crossings by rearranging the left side (leaving the right side fixed) is NP-complete.

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CR categories: F.2.2, G.2.2, I.3.5.

1. THE PROBLEM

For the purposes of this paper a ***bipartitioned graph*** $G=(L,R,E)$ consists of two sets L (the "left side") and R (the "right side") of vertices and a set $E \subseteq L x R$ of edges. Such graphs are commonly drawn with L on a vertical line on the left, and R on a vertical line on the right, as illustrated in Figure 1(a).

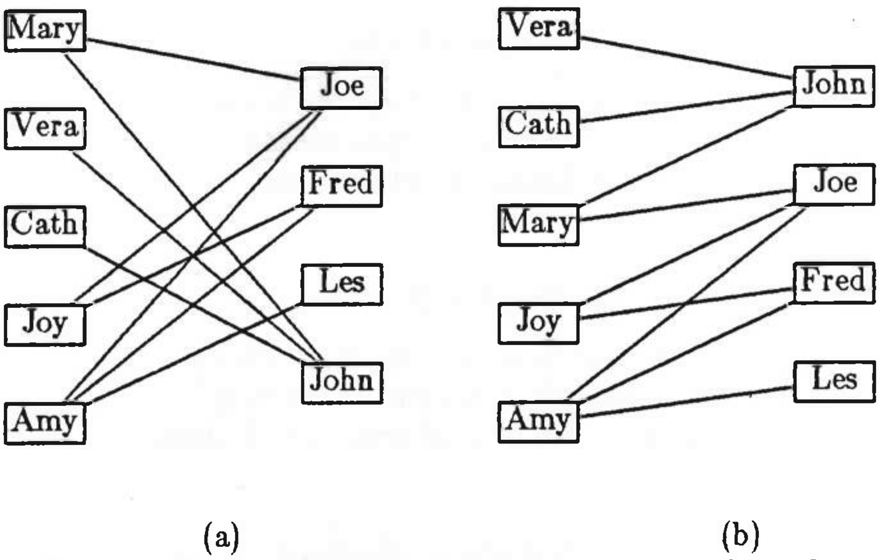

Figure 1: Two drawings of the same bipartitioned graph.

This drawing may be used as a graphic representation of a relation, or as a representation of the routing between two modules on a circuit board.

The appearance of such drawings may be enhanced by re-ordering the vertices on both sides to reduce the number of edge crossings. Reducing edge crossings is also important for circuit routing. Figure 1(a) is re-drawn in Figure 1(b) with fewer crossings.

Clearly, as long as edges are straight lines, the number of crossings depends only on the order of the vertices on the left and on the right. In section 2 of this paper we give a linear time algorithm for ordering the left and right sides so that no edges cross, if such an ordering is possible. The problem of ordering both sides to minimize the number of crossings is NP-complete (see Johnson (1982)). In section 3 of this paper we show that the problem of minimizing crossings by ordering just the left side, leaving the order of the right side fixed, is also NP-complete. This problem of finding a "left optimal drawing" arises in heuristics for drawing "layered networks" (see Eades and Kelly(1985)).

We need some terminology to make the problems precise. For the purposes of this paper an *ordering* of a finite set S is a one-one function $f:S \rightarrow \{1,2,\ldots,|S|\}$. A *drawing* (l,r) of a bipartitioned graph consists of a pair of orderings of the left and right sides respectively. A *crossing* in a drawing (l,r) of $G=(L,R,E)$ is a pair $(u,v),(w,x) \in E$ such that $l(u)<l(w)$ and $r(v)>r(x)$.

2. AN ALGORITHM FOR DRAWING WITHOUT CROSSINGS

We say that a bipartitioned graph is *crossing free* if it can be drawn without crossings. A crossing free network cannot contain a cycle, and Figure 2 shows that some trees are not crossing free.

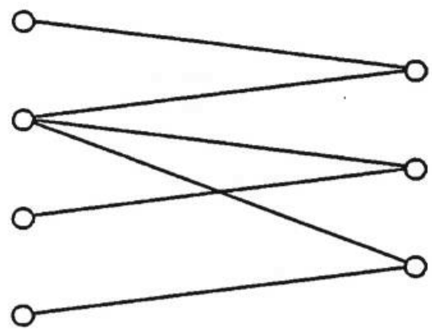

Figure 2: A tree which is not crossing free.

However, the class of crossing free networks can be recognized in linear time, as follows.

A graph is a *caterpillar* if it is a tree with one path (called the "main path") which contains all vertices of degree bigger than one. We claim that a bipartitioned graph is crossing free if and only if it is a collection of caterpillars. For suppose that $G=(L,R,E)$ is a connected bipartitioned graph and (l,r) is a drawing of G with no crossings. Then either the first vertex in L or the first vertex in R has degree 1. Removing this vertex gives (by induction) a caterpillar; replacing it does not destroy the caterpillar property. Conversely, a caterpillar may be drawn as illustrated in Figure 3. Each caterpillar has an *endpoint*, that is, a vertex u such that at most one of the vertices adjacent to u has degree more than one. From the endpoint, the main path can "wiggle" from top to bottom, with the "legs" in between the wiggles.

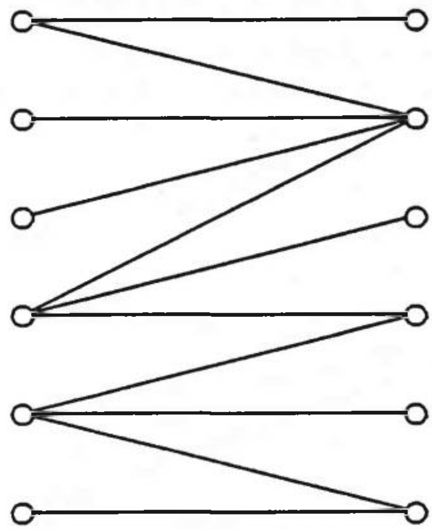

Figure 3: A drawing of a caterpillar.

Furthermore, by computing the degree of each vertex it is not difficult to recognize and draw caterpillars in linear time.

Crossing free algorithm

1. Compute the degree of each vertex; if the sum of the degrees is more than $2|V|-2$, then exit with a message "NOT A CATERPILLAR".

2. Divide the graph into connected components, and apply procedure *caterpillar* below to each component.

```
procedure caterpillar
   Find an endpoint u. If no such vertex exists,
   then exit with a message "NOT A CATERPILLAR".
   nextpos:=1;
   finished:=false;
   while not finished do
      position[u]:=nextpos;
      Let X denote the set of vertices attached to u which have
      no position assign to them as yet.
      If there is more than one vertex in X with degree bigger than one
      then
         exit with message "NOT A CATERPILLAR"
      else
         if there is a vertex w in X with degree bigger than one
         then u:=w
         else finished:=true;
         for each vertex v in X with degree 1 do position[v]:=nextpos
      nextpos:=nextpos+1
```

3. Output any two orderings l and r which are compatible with *position*.

3. THE NP-COMPLETENESS THEOREM

We consider the following problem:

Left Optimal Drawing (LOD)
Instance: Bipartitioned graph $G=(L,R,E)$, ordering r of R, integer B.
Question: Is there an ordering l of L so that (l,r) has at most B crossings?

Theorem: LOD is NP-complete.

It is clear that LOD is in NP. We give a problem in directed graphs and a transformation to LOD which proves that LOD is NP-complete. If D is a directed graph with vertex set V and arc set A, then a *feedback arc set* A' for D is a subset of A which contains at least one arc from each directed cycle of D. Note that A' is a feedback arc set if and only if the subgraph $(V,A-A')$ of D is acyclic. The following problem is NP-complete (see Garey and Johnson (1979) p192).

Feedback Arc Set (FAS)
Instance: Directed graph D, positive integer K.
Question: Does D have a feedback arc set of size at most K?

We give a polynomial transformation from FAS to LOD.

Suppose that $D=(V,A)$ and K form an instance of FAS. We construct a bipartitioned graph $G=(L,R,E)$ as follows. Let $L=V$. For each arc $a\in A$, we define a "clump" $C(a)=\{c_1(a),c_2(a),\ldots,c_6(a)\}$ of 6 vertices, and let R be the union over A of all the clumps.

For each $u\in V$ and $a\in A$ there are two edges joining u to $C(a)$. If $a=(u,v)$ for some $v\in V$ then u is joined to $c_1(a)$ and $c_5(a)$; if $a=(v,u)$ for some $v\in V$ then u is joined to $c_2(a)$ and $c_6(a)$. If u is not incident with a then u is joined to $c_3(a)$ and $c_4(a)$. The three possibilities are illustrated in Figure 4.

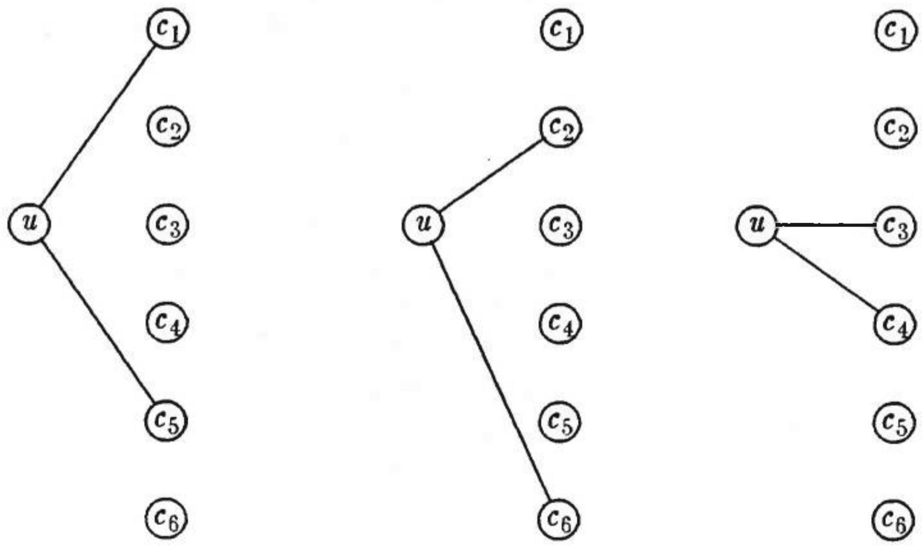

Figure 4: three possibilities.

Now let r be any ordering of R which keeps each clump together and in order, that is, so that $r(c_i(a))=r(c_1(a))+i-1$ for each $a\in A$, $i=1,2,\ldots,6$. Denote $|A|$ by α and $|V|$ by β, and let B be $4\binom{\alpha}{2}\binom{\beta}{2}+\alpha\binom{\beta-2}{2}+4\alpha(\beta-2)+\alpha+2K$.

An example of the transformation is in Figures 5 and 6.

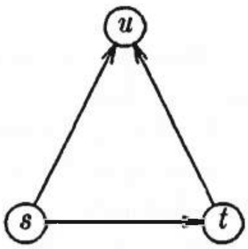

Figure 5: a directed graph D.

Lemma: If l is an ordering of L and A' denotes $\{(u,v)\in A: l(u)>l(v)\}$ then the number of crossings in the drawing (l,r) of G is $4\binom{\alpha}{2}\binom{\beta}{2}+\alpha\binom{\beta-2}{2}+4\alpha(\beta-2)+\alpha+2|A'|$.

Proof: If a and b are distinct arcs then, since every vertex u in L has two edges joining u to $C(a)$ and two to $C(b)$, there are $4\binom{\beta}{2}$ crossings between edges incident with $C(a)$ and edges incident with $C(b)$. This gives $4\binom{\alpha}{2}\binom{\beta}{2}$ crossings between edges from different clumps.

Now consider crossings between edges from the same clump $C(a)$. There are β-2 vertices in L which are incident with both $c_3(a)$ and $c_4(a)$, giving $\binom{\beta-2}{2}$ crossings between edges from these two vertices. If $a=(u,v)$, then each edge out of $c_3(a)$ and $c_4(a)$ crosses one of $(u,c_1(a))$ and $(u,c_5(a))$, and one of $(v,c_2(a))$ and $(v,c_6(a))$, giving another $4(\beta-2)$ crossings. The only remaining crossings are between $(u,c_1(a))$, $(u,c_5(a))$, $(v,c_2(a))$ and $(v,c_6(a))$. This is 1 if $l(u)<l(v)$, and 3 if $l(u)>l(v)$. Thus the total number of crossings between edges from $C(a)$ is $\binom{\beta-2}{2}+4(\beta-2)+1$ if $l(u)<l(v)$, and $\binom{\beta-2}{2}+4(\beta-2)+3$ if $l(u)>l(v)$. Summing over all clumps gives the result stated in the Lemma.

Now suppose that D has a feedback arc set A' of size at most K. Since $D'=(V,A-A')$ is acyclic we can obtain an ordering l of V so that $l(u)<l(v)$ whenever $(u,v)\in(A-A')$ by a topological sort. Since $|A'|\leq K$ and $A'=\{(u,v)\in A: l(u)>l(v)\}$, the Lemma implies that G has at most B crossings.

Conversely, suppose that l is an ordering of L such that G has at most B crossings. It follows from the Lemma that if $A'=\{(u,v)\in A: l(u)>l(v)\}$ then $|A'|\leq K$. Further, $D'=(V,A-A')$ must be acyclic and so A' is a feedback arc set. This completes the proof.

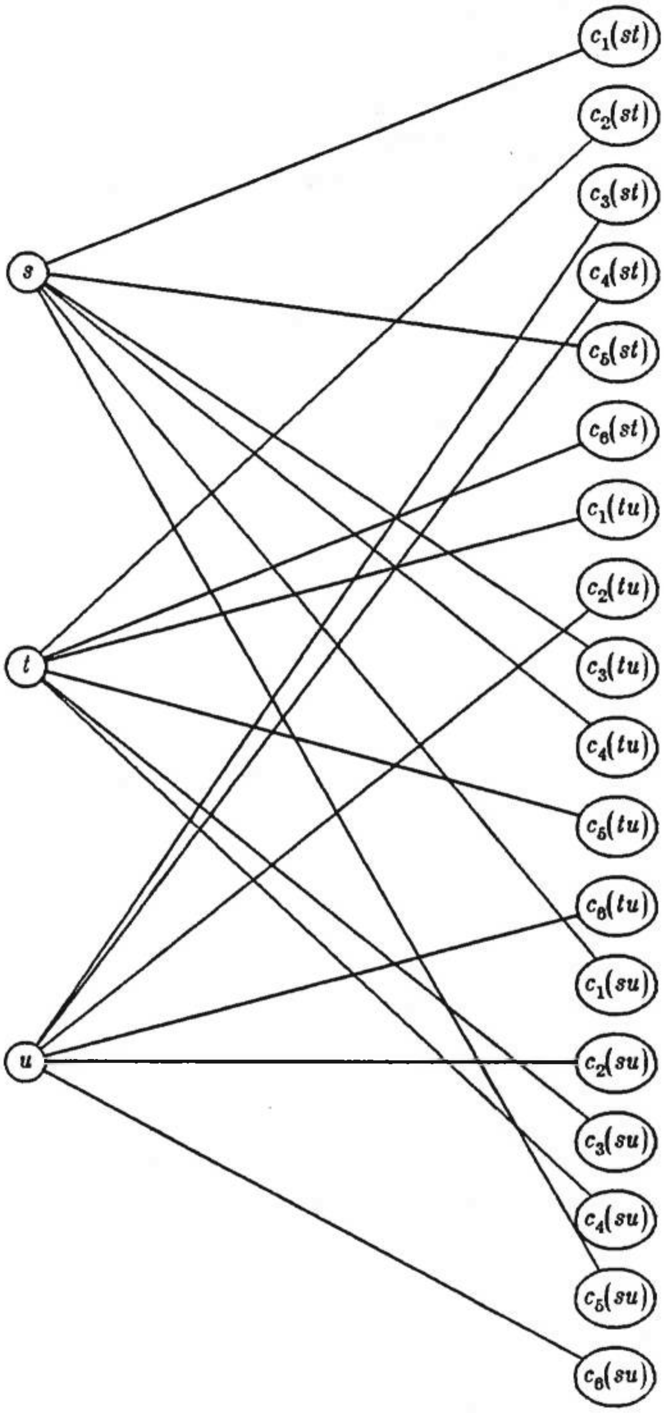

Figure 6: The bipartitioned graph G corresponding to the directed graph D in Figure 5.

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