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Circular designs balanced for neighbours at distances one and two

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SUMMARY

We define three types of neighbour-balanced designs for experiments where the units are arranged in a circle or single line in space or time. The designs are balanced with respect to neighbours at distance one and at distance two. The variants come from allowing or forbidding self-neighbours, and from considering neighbours to be directed or undirected. For two of the variants, we give a method of constructing a design for all values of the number of treatments, except for some small values where it is impossible. In the third case, we give a partial solution that covers all sizes likely to be used in practice.

Some key words: Border plot; Circular design; Eulerian trail; Latin square; Neighbour design; Perfect cycle system; Quasigroup; Universal sequence.

1. A DESIGN PROBLEM

The following experiment was brought to our attention by R. M. Cormack, University of St Andrews. A marine biologist wanted to compare five genotypes of bryozoan by suspending them in sea water around the circumference of a cylindrical tank. Each genotype was replicated five times, so that altogether 25 items were suspended in the tank: see Bayer & Todd (1996).

When the experiment was being planned, there was a suggestion that neighbouring genotypes might interfere with each other. The model assumed was that the response y_i at site i satisfies

$$y_i = \lambda_{\tau(i-1)} + \delta_{\tau(i)} + \rho_{\tau(i+1)} + \varepsilon_i, \tag{1}$$

(1, 1, 3, 4, 3, 0, 0, 1, 0, 2, 2, 0, 3, 3, 1, 2, 1, 4, 0, 4, 4, 2, 3, 2, 4)

Fig. 1. Circular design for five treatments in 25 plots: it is strongly balanced for neighbours at distances one and two.

where $\tau(i)$ denotes the genotype at site i, the ε_i are independent random variables with mean zero and common variance σ^2, and arithmetic on the site labels is performed modulo 25. Here δ_j is the direct effect of genotype j, while λ_j and ρ_j are the left- and right-neighbour effects of genotype j, respectively, which are not assumed to be the same.

Under model (1), the sums of the variances of the estimates of the difference between two δ-parameters or two λ-parameters or two ρ-parameters are all minimized when (i) each ordered pair of items occurs just once as neighbours around the circumference of the tank, and (ii) each ordered pair of items occurs just once with a single item between them. Conditions (i) and (ii) are called strong neighbour balance at distances one and two respectively. Designs that minimize these sums of variances are called optimal.

A design with these properties is shown in Fig. 1. The parentheses indicate that the sequence is to be interpreted as a circle. Later, we use square brackets to indicate a sequence that is simply a line. From now on, sequence means circular sequence unless otherwise stated.

A design like the one in Fig. 1 can also be used for n treatments in a long line of $n^2 + 2$ plots. The two end plots and the n^2 inner plots are all used for the experiment but only those measurements on the inner plots are analysed. In agriculture and forestry, such end plots are known as guard plots or border plots: see Azaïs et al. (1993). To create the linear design, cut the circular design open between any pair of items and straighten it out: this gives the inner plots. The treatment on each border plot is the same as the treatment on the inner plot at the opposite end of the design.

Jenkyn & Dyke (1985) argued that model (1) is appropriate for experiments on fungicides or pesticides when plots are in a single line. Spores from plots with less effective treatments may spread to their neighbours, and changeable wind patterns imply that neighbour effects from different sides are not the same.

When there are n^2 inner plots, conditions (i) and (ii) together are equivalent to the following condition: (iii) among the triples of the form $[\tau(i-1), \tau(i), \tau(i+1)]$, each ordered pair of treatments occurs once in positions 1 and 2, once in positions 1 and 3, and once in positions 2 and 3. Such designs are suitable whenever model (1) can be assumed. There are $3n - 2$ linear parameters to be estimated from n^2 measurements, so we require $3n - 2 \leqslant n^2$, which is true for all positive integers n.

Another use of such designs is for cross-over trials on a single subject. If it is assumed that the response y_i in period i is affected not only by the direct effect of the treatment applied in period i but also by the carry-over effects of the treatments applied in the two previous periods, then

$$y_i = \kappa_{\tau(i-2)} + \lambda_{\tau(i-1)} + \delta_{\tau(i)} + \varepsilon_i .$$

In such an experiment, the first two periods are used as pre-periods and measurements are made on the remaining n^2 periods. Again, condition (iii) ensures minimum variance of the estimators.

The problem is: how do we construct designs satisfying condition (iii)? In the next section, we show that our designs are equivalent to certain sorts of quasigroups and to certain sorts of trails in graphs. After that, we introduce two variants on the design, both with no self-neighbours: direction is relevant in one variant but not in the other. Both of these appear to be useful. In § § 5–6 we give complete solutions to the problem of constructing both the variant types of design, before

(a)

$\circ$	0	1	2	3	4
0	1	0	2	3	4
1	2	3	1	4	0
2	3	4	0	2	1
3	0	2	4	1	3
4	4	1	3	0	2

(b)

$\circ$	0	1	2	3	4
0	1	2	3	0	4
1	0	3	4	2	1
2	2	1	0	4	3
3	3	4	2	1	0
4	4	0	1	3	2

Fig. 2. Two quasigroups: (a) corresponds to the circular design in Fig. 1, while its transpose, (b), is not Eulerian.

returning to the original case in § 7 and giving a partial solution. Some related open problems are discussed in § 8.

2. Combinatorially equivalent problems

Condition (iii) turns up in another context. Applied to the triples of the form [row, column, symbol], it is precisely the definition of a Latin square of order n. In our application, the neighbour treatments are from the same set as the direct treatments, so we have a Latin square whose rows and columns are labelled by the set of symbols used in the body of the square. Technically, such a square is called a quasigroup: see Street & Street (1987, Ch. 5). The quasigroup operation $\circ$ is defined by

$$a \circ b = \text{symbol in row } a \text{ and column } b.$$

In the circular design, each triple should have the form $[a, b, a \circ b]$. The quasigroup given by the design in Fig. 1 is in Fig. 2(a).

Conversely, we can start with any ordered pair $[x, y]$ and successively build the design from the quasigroup as

$$x, \quad y, \quad x \circ y, \quad y \circ (x \circ y), \quad (x \circ y) \circ (y \circ (x \circ y)), \quad \ldots .$$

The Latin property ensures that the sequence cannot return to any pair in the sequence before the pair at the start. However, for the majority of quasigroups this sequence comes back to the start in fewer than n^2 steps. For example, the quasigroup in Fig. 2(b) gives the circular sequences (0, 0, 1, 2, 4, 3, 3, 1, 4, 1), (0, 2, 3, 4, 0, 4, 4, 2, 1, 1, 3, 2, 2) and (0, 3), of lengths 10, 13 and 2, rather than a single circular sequence of length 25. In analogy with Eulerian trails in graphs, we call a quasigroup Eulerian if the above construction gives a single circle of length n^2, and we also call this circular sequence Eulerian. We seek an Eulerian quasigroup for each value of n. The two quasigroups in Fig. 2 demonstrate that interchanging rows and columns need not preserve the Eulerian property. However, if we interchange rows and symbols then each column is replaced by its inverse permutation. This simply reverses the circular sequence associated with an Eulerian quasigroup, thereby preserving the property.

Our problem can also be considered to be one in graph theory. The circular sequence in Fig. 1 gives a way of traversing the edges in the complete directed graph $\vec{K}_5^\circ$ on five vertices with a loop at each vertex. Condition (i) is equivalent to this sequence being an Eulerian trail; condition (ii) imposes an extra constraint. In the graph-theoretical language used by Bryant & Adams (1993), these conditions mean that the circle of items is 1-perfect and 2-perfect, although those terms are usually applied to systems of cycles in which items cannot be repeated within a single cycle.

3. No self-neighbours

Sometimes, practical considerations make it undesirable to have the same treatment on neighbouring plots. See Dyke & Shelley (1976), who assumed model (1) and used $n(n-1)^2+2$ plots to give all ordered triples with no self-neighbours. Slightly different are the serially balanced sequences of Nair (1967), which have $n(n-1)(n-2)+2$ plots with all ordered triples of three distinct treatments. However, if condition (i) is modified to exclude self-neighbours, then it suffices to use $n(n-1)$ inner plots. Estimability of all the differences requires $3n-2 \leqslant n(n-1)$, which is true when $n \geqslant 4$. Then the incidence of direct treatments with either left- or right-neighbour treatments is that of a symmetric balanced incomplete-block design for n treatments in blocks of size $n-1$. In the absence of left-neighbour effects, this would produce an optimal design for the direct and right-neighbour effects, as shown in a 2000 University of London PhD thesis by C. Lewis.

For similar balance between left- and right-neighbour effects, all that is needed is that every right-neighbour occurs once with all but one of the left-neighbours. However, pairwise balance among all pairs from three factors is not sufficient to give overall balance in the sense that estimators of all differences between effects of the same sort have the same variance: see Bailey (1999) and Preece (1966, 1976, 1988). To achieve overall balance, it is necessary for the missing pairs at distance two to be the same as the missing pairs at distance one. Therefore we also modify condition (ii) to exclude the same treatment at distance two.

Druilhet (1999) considered designs in circular blocks of size n or $n-1$, and showed that designs satisfying the modified conditions (i) and (ii) are optimal for estimating direct effects and neighbour effects under model (1) among designs with no self-neighbours. The proof also holds for our situation of an equireplicate design in a single large block.

Fortunately, the statistical desiderata match the combinatorial ones. In the quasigroup, suppose that $a \circ a = b$. If the sequence does not contain the pair $[a, a]$, then it cannot have the pair $[a, b]$ either. So we must insist that $a \circ a = a$ for all symbols a. A quasigroup is said to be idempotent if it satisfies this condition. Any collection of circular sequences built from such a quasigroup has all distinct ordered pairs at distances one and two but no self-pairs at distance one or two.

In graph-theoretical terms, we now seek an Eulerian trail of the complete directed loopless graph $\vec{K}_n$ on n vertices subject to the constraint imposed by the modified condition (ii).

Let us call a circular sequence, or equivalent quasigroup, idempotent Eulerian if it contains n treatments in $n(n-1)$ plots, with every ordered pair of distinct elements occurring once at distances one and two. Of course, such a sequence can also be opened out into a linear sequence for $n(n-1)+2$ plots. A similar notion, treated from a combinatorial perspective, is the idea of a universal sequence: see, for example, Brockman et al. (2010).

4. Undirectional neighbour effects

Finally, we consider designs for experiments where the effect of the neighbouring treatment is the same whether it is from the left or the right. In agricultural experiments this is a reasonable assumption if the neighbour effect is caused simply by proximity, such as competition for resources (Kempton & Lockwood, 1984) or roots of plants reaching into nearby plots (Welham et al., 1996). To improve control of heterogeneity, Philippeau et al. (1996) and David et al. (2001) recommend that a variety trial should use long thin plots, each consisting of such a small number of rows of plants along the plot length that it is wasteful to disregard the data from the exterior rows. Now the model is

$$y_i = \lambda_{\tau(i-1)} + \delta_{\tau(i)} + \lambda_{\tau(i+1)} + \varepsilon_i . \tag{2}$$

(0, 1, 2, 0, 3, 4, 1, 5, 6, 0, 5, 3, 1, 6, 2, 4, 5, 2, 3, 6, 4)

Fig. 3. Circular design for seven treatments in 21 plots: each pair of distinct treatments are neighbours at distances one and two exactly once.

The previous designs are still suitable, but we could consider saving resources by using a design in which each treatment is a neighbour of every other treatment just once and is at distance two from each other treatment just once. This is impossible if n is even, because each treatment has an even number of neighbours overall, but it may be possible if n is odd. Such a design would use a circle of $n(n-1)/2$ plots, or a long line of $n(n-1)/2$ inner plots with two border plots. Figure 3 shows such a design for $n = 7$.

It seems plausible that such a design is optimal among designs of this size. Proof would require an extension of the work by Druilhet (1999).

We call such a circular sequence semi-Eulerian. A semi-Eulerian sequence can be used to construct an idempotent quasigroup by defining $a \circ b = c$ and $c \circ b = a$ for every subsequence $[a, b, c]$. This quasigroup is equal to its conjugate obtained by interchanging rows and symbols.

There seems to be no statistical reason to ban self-neighbours from a design for undirectional neighbour effects. However, if we allow self-neighbours at distance one, then we must also allow self-neighbours at distance two, for there are now $n(n+1)/2$ plots for $n(n-1)/2$ unordered pairs and n self-pairs. Now, any triple of the form $[a, b, a]$ gives the unordered pair $\{a, b\}$ twice at distance one, so it is impossible to have self-neighbours in a design in which each unordered pair occurs exactly once at distance one and once at distance two.

There are now $2n - 1$ independent parameters to estimate, so we need $2n - 1 \leqslant n(n-1)/2$, which is true if $n \geqslant 5$.

5. Solution with no self-neighbours

In this section we solve the existence question for idempotent Eulerian sequences. Let $m = n - 1$. Throughout this section, the treatments are labelled by the integers modulo m, together with ∞. All calculations are modulo m.

Proposition 1. *Let $[a_1, a_2, \ldots, a_{m-1}]$ be a linear arrangement of the nonzero integers modulo m. Put $b_i = a_i + a_{i+1}$ for $i = 1, \ldots, m-2$. If there exists an arrangement in which* (a) *the b_i are all distinct and nonzero,* (b) *the nonzero element b' missing from $\{b_1, \ldots, b_{m-2}\}$ is 1 if m is odd and is $m/2 + 1$ if m is even, and* (c) *$a_{m-1} + a_1 = 1$, then there is an idempotent Eulerian circular sequence of order n.*

Proof. Put $c_0 = 0$ and, for $i = 1, \ldots, m-1$, put $c_i = c_{i-1} + a_i$. Then c_{m-1} is the sum of the nonzero integers modulo m, which is 0 if m is odd and $m/2$ if m is even; in both cases $c_{m-1} = -c_{m-1}$. Moreover, $a_1 + a_{m-1} + b_1 + \cdots + b_{m-2}$ is twice this sum, which is zero in both cases, so conditions (a)–(c) show that $1 = a_1 + a_{m-1} = b' - c_{m-1}$.

Write the linear sequence $[\infty, c_0, c_1, \ldots, c_{m-1}]$ as the first row of an $m \times n$ matrix, and develop subsequent rows by adding 1 modulo m, with the convention that $\infty + 1 = \infty$. Put the rows one after another to give the desired circular sequence.

All columns except the first contain each integer modulo m just once. It is therefore clear that ∞ is preceded and followed by every other treatment just once at both distances one and two.

If x and y are distinct integers modulo m, then there is some i such that $y - x = a_i$. The column with c_{i-1} at the top has x in a unique row; the next element in that row is $c_i + (x - c_{i-1}) = x + a_i = y$.

$(\infty, 0, 4, 2, 3, 0, \infty, 1, 0, 3, 4, 1, \infty, 2, 1, 4, 0, 2, \infty, 3, 2, 0, 1, 3, \infty, 4, 3, 1, 2, 4)$

Fig. 4. An idempotent Eulerian circular sequence of order 6.

If there is some j such that $y - x = b_j$, then a similar argument shows that there is a unique row in which y occurs two places to the right of x. Otherwise, $y - x = b' = c_{m-1} + 1$. Now, there is a unique row in which x occurs in the final column; the second element of the next row is $c_0 + (x - c_{m-1}) + 1 = y$. □

Write a and b for the linear sequences $[a_1, \ldots, a_{m-1}]$ and $[b_1, \ldots, b_{m-2}]$ respectively.

For example, let $n = 6$, so that $m = 5$. Put $a = [4, 3, 1, 2]$. Then $a_1 + a_4 = 1$, $b = [2, 4, 3]$, and $b' = 1$. The matrix is

$$\begin{bmatrix} \infty & 0 & 4 & 2 & 3 & 0 \\ \infty & 1 & 0 & 3 & 4 & 1 \\ \infty & 2 & 1 & 4 & 0 & 2 \\ \infty & 3 & 2 & 0 & 1 & 3 \\ \infty & 4 & 3 & 1 & 2 & 4 \end{bmatrix}$$

and the circular sequence is in Fig. 4.

Any idempotent Eulerian sequence gives an idempotent Eulerian quasigroup. For a sequence constructed by the method in Proposition 1, the quasigroup has a cyclic automorphism of order m and so is bordered diagonally cyclic in the sense of Wanless (2004a) and Bryant et al. (2009), who give a full analysis of such quasigroups and their many applications in design theory.

Theorem 1. *If $n \geqslant 6$, then there exists an idempotent Eulerian circular sequence of order n.*

Proof. In view of Proposition 1, it suffices to give a linear sequence a of integers modulo m with the right properties. We divide the values of n into five different cases. In each case, we use a vertical bar to show a change in pattern in the sequence.

Case 1. Suppose that $n = 4k$ and $k \geqslant 2$: then $m = 4k - 1$. Put

$$\begin{aligned} a = [&2k-2, 2k-1, 2k-4, 2k-3, \ldots, 2, 3 \mid 1 \mid \\ &4k-3, 4k-2, 4k-5, 4k-4, \ldots, 2k+3, 2k+4 \mid 2k+1, 2k, 2k+2]. \end{aligned}$$

Then $b = [4k-3, 4k-5, \ldots, 7, 5 \mid 4 \mid 4k-2 \mid 4k-4, 4k-6, \ldots, 8 \mid 6 \mid 2, 3]$. Moreover, $a_1 + a_{m-1} = 1$ and $b' = 1$.

Case 2. Suppose that $n = 4k + 1$ and k is even: then $m = 4k$. Put

$$\begin{aligned} a = [&k+1, 3k+1, 3k+2, k+2, k+3, 3k+3, 3k+4, k+4, \ldots, \\ &2k-2, 2k-1, 4k-1 \mid 2k+1, 2k, 2, 1, 2k+3, 2k+2, 4, 3, \ldots, \\ &3k-1, 3k-2, k, k-1 \mid 3k]. \end{aligned}$$

Then

$$b = [2, 2k+3, 4, 2k+5, \ldots, 4k-3, 2k-2 \mid 2k \mid 1, 2k+3, 3, 2k+4, \ldots,$$
$$2k-3, 4k-2, 2k-1 \mid 4k-1],\ a_1 + a_{m-1} = 1 \text{ and } b' = 2k+1.$$

Case 3. Suppose that $n = 4k+1$, k is odd and $k \geqslant 3$: then $m = 4k$. Put

$$a = [3k+1, k+1, k+2, 3k+2, 3k+3, k+3, k+4, \ldots,$$
$$2k-2, 2k-1, 4k-1 \mid 2k+1, 2k, 2, 1, 2k+3, 2k+2, 4, 3, \ldots,$$
$$k-1, k-2, 3k, 3k-1 \mid k].$$

Then

$$b = [2, 2k+3, 4, 2k+5, 6, 2k+7, \ldots, 4k-3, 2k-2 \mid 2k \mid 1, 2k+2, 3, 2k+4, \ldots,$$
$$2k-3, 4k-2, 2k-1 \mid 4k-1],\ a_1 + a_{m-1} = 1 \text{ and } b' = 2k+1.$$

Case 4. If $n = 4k+2$, then $m = 4k+1$. Put

$$a = [2k, 2k-1, \ldots, 2, 1 \mid 4k-1, 4k, 4k-3, 4k-2, \ldots, 2k+1, 2k+2].$$

Then $b = [4k-1, 4k-3, \ldots, 5, 3 \mid 4k \mid 4k-2, 4k-4, 4k-6, \ldots, 2]$, $a_1 + a_{m-1} = 1$ and $b' = 1$.

Case 5. If $n = 4k+3$, then $m = 4k+2$. Put

$$a = [2k+2, 1, 2k+3, 2, 2k+4, 3, \ldots, 3k+1, k \mid$$
$$k+1, 3k+2, k+2, 3k+3, \ldots, 2k, 4k+1, 2k+1].$$

Then $b = [2k+3, 2k+4, \ldots, 4k, 4k+1 \mid 2k+1 \mid 1, 2, \ldots, 2k]$, $a_1 + a_{m-1} = 1$ and $b' = 2k+2$. □

An exhaustive computer search showed that there is no idempotent Eulerian quasigroup of order five or less, whether or not made by the construction in Proposition 1. Hence Theorem 1 constructs an idempotent Eulerian quasigroup for every possible order.

The idempotent Eulerian sequences constructed above are balanced at distances one and two. If we use $a = [2, 9, 1, 13, 6, 7, 5, 3, 14, 4, 11, 10, 12, 8, 15]$ with $m = 16$ in Proposition 1, then we produce an example of order $n = 17$ that achieves balance at distances one, two and three. It remains an open question as to what other examples of this type exist. It is also not clear whether an idempotent Eulerian sequence can be balanced at greater distances.

6. Solution in the undirected case

In this section we construct a semi-Eulerian sequence for all odd values of $n \geqslant 7$.

Proposition 2. *Let $n = 2r+1$ and let $(a_1, \ldots, a_r)$ be a circular sequence of integers modulo n. Put $b_i = a_i + a_{i+1}$ for $1 \leqslant i \leqslant r-1$ and $b_r = a_r + a_1$. Let $c = \sum_{i=1}^{r} a_i$. If the $\pm a_i$ are all different modulo n and the $\pm b_i$ are all different modulo n and c is coprime to n, then there is a semi-Eulerian sequence of order n.*

Proof. The desired circular sequence is

$$(a_1,\ \ a_1+a_2,\ \ a_1+a_2+a_3,\ \ \ldots,\ \ c,\ \ c+a_1,\ \ c+a_1+a_2,\ \ \ldots,\ \ 2c,\ \ \ldots,\ \ -a_r,\ \ 0).$$

If c is coprime to n, this gives a circular sequence of length nr where each circular subsequence of entries at distance r contains all the integers modulo n just once. Suppose that x and y are distinct integers modulo n. The condition on the a_i implies that none of them is zero, so there is some i such that $y-x=\pm a_i$. If $y-x=a_i$, then the ordered pair $[x, y]$ occurs in the circular sequence; otherwise the ordered pair $[y, x]$ occurs. The condition on the b_i gives the same property at distance two. □

THEOREM 2. *There exists a semi-Eulerian sequence of order n for all odd n with $n \geqslant 7$.*

Proof. We gave an example for $n=7$ in Fig. 3. For larger n we display a circular sequence satisfying the hypotheses of Proposition 2. For $n=11$ the sequence $a=(1, 2, 3, 7, 6)$ suffices. For $n=9$ and $n \geqslant 13$ we treat different cases depending on the residue of n modulo 12.

Case 1. If $n=12k+1$, we take

$$a=\big(1, \ldots, 2k \mid 6k, \ldots, 5k+1 \mid -5k, \ldots, -(2k+1)\big),$$

so that $c=-3k^2$. It is clear that $\gcd(c, n)=1$ since $n \equiv 1 \mod 3$ and $n \equiv 1 \mod k$.

Case 2. If $n=12k+3$ and $k \not\equiv 2 \mod 3$, then we can take

$$a=\big(1, \ldots, k \mid -(k+1), \ldots, -2k \mid 4k+1, \ldots, 2k+1) \mid -(6k+1), \ldots, -(4k+2)\big),$$

so that $c=-5k^2+2k+1$. Now $9=48c+(20k-13)n$ so $\gcd(c, n)$ divides 9, but $c \equiv 1 \mod 3$, so $\gcd(c, n)=1$.

Case 3. If $n=12k+3$ and $k \not\equiv 1 \mod 3$, then we can take

$$a=\big(1, \ldots, 2k \mid 6k+1, \ldots, 5k+2 \mid -(5k+1), \ldots, -(4k+1) \mid 2k+1, \ldots, 4k\big),$$

so that $c=9k^2-2k-1$. To see that $\gcd(c, n)=1$ we note that $3=48c-(36k-17)n$ so $\gcd(c, n)$ divides 3, and yet $c \not\equiv 0 \mod 3$.

Case 4. If $n=12k+5$, then we take

$$a=\big(1, \ldots, k \mid -3k, \ldots, -(2k+1) \mid 2k, \ldots, k+1 \mid 5k+3, \ldots, 6k+2 \\ \mid -(4k+1), \ldots, -(3k+1) \mid -(5k+2), \ldots, -(4k+2)\big).$$

Then $c=-3k^2-8k-3$, so $3=-16c-(4k+9)n$ but $n \equiv 2 \mod 3$, so $\gcd(c, n)=1$.

Case 5. If $n=12k+7$, then we take

$$a=\big(1, \ldots, k \mid 3k+1, \ldots, 2k+1 \mid -2k, \ldots, -(k+1) \mid 3k+2, \ldots, 6k+3\big).$$

Now $c=15k^2+20k+6$, so $9=-16c+(20k+15)n$ and $n \equiv 1 \mod 3$, so $\gcd(c, n)=1$.

Case 6. If $n = 12k + 9$, then we take

$$a = \big(1, \ldots, 4k + 2 \mid -(6k + 4), \ldots, -(5k + 4) \mid 5k + 3, \ldots, 4k + 3\big).$$

Now $c = 7k^2 + 8k + 2$, so $3 = -48c + (28k + 11)n$ and $c \not\equiv 0 \mod 3$, so $\gcd(c, n) = 1$.

Case 7. If $n = 12k + 11$, then we take

$$\begin{aligned} a = \big(&1, \ldots, k + 1 \mid 5k + 5, \ldots, 4k + 4 \mid -(4k + 3), \ldots, -(3k + 3) \\ &\mid 5k + 6, \ldots, 6k + 5 \mid -(2k + 2), \ldots, -(3k + 2) \mid -(k + 2), \ldots, -(2k + 1)\big), \end{aligned}$$

so that $c = 3k^2 + 8k + 5$. Now $3 = 16c - (4k + 7)n$ and $n \equiv 2 \mod 3$, so $\gcd(c, n) = 1$.

The above cases together cover all possibilities. □

As we have noted earlier, semi-Eulerian sequences can exist for odd n only. By exhaustive computation, it is simple to confirm there is no example for $n = 5$. Thus Theorem 2 solves the existence question for semi-Eulerian sequences in the estimable case when $n \geqslant 5$. The sequences (0) and (0, 1, 2) provide simple, although arguably degenerate, examples of order $n = 1$ and $n = 3$ that satisfy the combinatorial constraints, though not the statistical ones.

7. Eulerian quasigroups

In this penultimate section we give a partial solution to the existence question for Eulerian quasigroups. For $n \in \{2, 3, 4\}$ it is quickly established by exhaustive search that there are none.

Conjecture 1. There exists an Eulerian quasigroup of every order $n \geqslant 5$.

We will confirm Conjecture 1 for all small n. In particular:

Theorem 3. *If n is a counterexample to Conjecture* 1*, then n is divisible by a prime power exceeding* 1000.

Given two quasigroups $(Q_1, \cdot)$ and $(Q_2, \circ)$ of orders n and m, we can form their tensor product $Q_1 \otimes Q_2$, which is a quasigroup of order nm. The underlying set is $Q_1 \times Q_2$ and the operation $\bullet$ is defined by $(a, x) \bullet (b, y) = (a{\cdot}b, x \circ y)$ for a, b in Q_1 and x, y in Q_2.

Proposition 3. *If Q_1 and Q_2 are Eulerian quasigroups of orders n and m, and n is coprime to m, then $Q_1 \otimes Q_2$ is an Eulerian quasigroup.*

Proof. In the sequence

$$(a, x), \quad (b, y), \quad (a{\cdot}b,\ x \circ y), \quad \big(b{\cdot}(a{\cdot}b),\ y \circ (x \circ y)\big), \quad \ldots,$$

adjacent pairs of first coordinates repeat every n^2 steps but not earlier, while adjacent pairs of second coordinates repeat every m^2 steps but not earlier. If m is coprime to n, then adjacent pairs in both coordinates do not repeat until n^2m^2 steps. □

We build Eulerian quasigroups by exploiting the fact that the Eulerian property is not preserved by isotopy, that is, when the rows, columns, and/or symbols are permuted. Indeed, every Eulerian quasigroup of order at least two is isotopic to a non-Eulerian quasigroup since the symbols can always be permuted so that symbol 0 is in the cell in row 0 and column 0.

Table 1. *Permutations for odd prime powers q in the range $5 < q < 100$*

q	permutation	q	permutation	q	permutation	q	permutation	q	permutation
7	$\pi_0\pi_{3,1}$	23	$\pi_0\pi_{15,3}$	41	$\pi_0\pi_{3,5}\pi_{30,5}$	61	$\pi_0\pi_{37,2}$	83	$\pi_0\pi_{15,3}$
9	$\pi_0\pi_{4,2}$	25	$\pi_0\pi_{3,1}\pi_{6,9}$	43	$\pi_0\pi_{3,1}\pi_{27,2}$	67	$\pi_0\pi_{3,5}\pi_{19,2}$	89	$\pi_0\pi_{34,10}$
11	$\pi_0\pi_{5,1}$	27	$\pi_0\pi_{5,1}$	47	$\pi_0\pi_{11,13}$	71	$\pi_0\pi_{4,15}$	97	$\pi_0\pi_{46,4}$
13	$\pi_0\pi_{8,2}$	29	$\pi_0\pi_{5,4}$	49	$\pi_0\pi_{16,10}$	73	$\pi_0\pi_{3,1}\pi_{20,25}$		
17	$\pi_0\pi_{3,2}\pi_{8,4}$	31	$\pi_0\pi_{12,1}$	53	$\pi_0\pi_{32,2}$	79	$\pi_0\pi_{29,5}$		
19	$\pi_0\pi_{3,1}\pi_{6,1}\pi_{14,1}$	37	$\pi_0\pi_{3,3}\pi_{18,7}$	59	$\pi_0\pi_{27,1}$	81	$\pi_0\pi_{47,2}$		

$\odot$	0	1	2	3	4	5	6
0	1	2	0	4	3	5	6
1	2	0	4	3	5	6	1
2	0	4	3	5	6	1	2
3	4	3	5	6	1	2	0
4	3	5	6	1	2	0	4
5	5	6	1	2	0	4	3
6	6	1	2	0	4	3	5

Fig. 5. Eulerian quasigroup corresponding to a circular design for seven treatments in 49 plots, balanced for neighbours at distances one and two.

If n is odd, it seems that an effective approach to constructing an Eulerian quasigroup is to permute the symbols in the addition table of the cyclic group $(\mathbb{Z}_n, +)$. In other words, we choose a permutation $\pi : \mathbb{Z}_n \to \mathbb{Z}_n$ and define a new quasigroup $(\mathbb{Z}_n, \odot)$ by the rule $x \odot y = \pi(x + y)$, for all $x, y \in \mathbb{Z}_n$. For most small odd orders there are numerous permutations that produce an Eulerian quasigroup, and we have the luxury of choosing one that can be specified compactly.

Define $\pi_{i,j}$ to be the permutation with cycles

$$(i, i + 1)(i + 2, i + 3) \cdots (i + 2j - 2, i + 2j - 1);$$

in other words j consecutive transpositions starting at i. Also, let π_0 denote the cycle $(0, 1, 2)$. For each odd prime power q with $5 < q < 1000$, we found a permutation π for which the above rule gives an Eulerian quasigroup. Those for $q < 100$ are shown in Table 1, whose extension to $q < 1000$ is in the Supplementary Material. For $n = 5$ an example of an Eulerian quasigroup is in Fig. 2(a). Hence we have one for every odd prime power below 1000, except for $n = 3$.

For example, for $q = 7$ Table 1 gives the permutation $\pi_0\pi_{3,1}$, which is $(0, 1, 2)(3, 4)$. This gives the Eulerian quasigroup in Fig. 5, which in turn gives a circular design for seven treatments in 49 plots, balanced for neighbours at distances one and two.

To work around the nonexistence of an Eulerian quasigroup of order 3, we found a permutation of $\mathbb{Z}_{3q}$ for each prime power q coprime to 6 with $q < 1000$. Table 2 gives these permutations for $q < 100$, and the full list is in the Supplementary Material. The interpretation is analogous to Table 1. Combining these results with Proposition 3 proves Theorem 3 for odd n.

When n is even, there is an obstacle that prevents us from using the same method that we used for the odd orders. To describe the obstacle we need some theory from Wanless (2004b).

Suppose that $\circ$ is a right-cancellative binary operation on a set Q, which means that it satisfies the law $a \circ c = b \circ c \Rightarrow a = b$. For each element $y \in Q$ we can define a permutation $\phi_y : Q \to Q$ by $x \mapsto x \circ y$. Let sgn denote the parity homomorphism from the symmetric group to $\mathbb{Z}_2$, so that $\mathrm{sgn}(\pi) = 0$ if π is an even permutation and $\mathrm{sgn}(\pi) = 1$ if π is an odd permutation. The binary value $\sum_{y \in Q} \mathrm{sgn}(\phi_y)$ is known as the column parity of $(Q, \circ)$.

Table 2. *Permutations for order $n = 3q < 300$ where q is a power of a prime $p > 3$*

n	permutation	n	permutation	n	permutation	n	permutation	n	permutation
15	$\pi_0\pi_{12,1}$	57	$\pi_0\pi_{3,1}\pi_{19,3}$	111	$\pi_0\pi_{3,1}\pi_{31,8}$	159	$\pi_0\pi_{86,13}$	219	$\pi_0\pi_{3,1}\pi_{26,22}$
21	$\pi_0\pi_{7,1}\pi_{12,1}$	69	$\pi_0\pi_{16,20}$	123	$\pi_0\pi_{3,1}\pi_{37,32}$	177	$\pi_0\pi_{120,4}$	237	$\pi_0\pi_{191,20}$
33	$\pi_0\pi_{3,1}\pi_{22,3}$	75	$\pi_0\pi_{3,2}\pi_{14,11}$	129	$\pi_0\pi_{12,58}$	183	$\pi_0\pi_{3,2}\pi_{19,51}$	249	$\pi_0\pi_{6,10}$
39	$\pi_0\pi_{9,3}$	87	$\pi_0\pi_{3,4}\pi_{14,11}$	141	$\pi_0\pi_{3,1}\pi_{68,9}$	201	$\pi_0\pi_{176,6}$	267	$\pi_0\pi_{3,1}\pi_{43,24}$
51	$\pi_0\pi_{3,2}\pi_{30,9}$	93	$\pi_0\pi_{12,28}$	147	$\pi_0\pi_{84,21}$	213	$\pi_0\pi_{155,4}$	291	$\pi_0\pi_{3,2}\pi_{32,7}$

THEOREM 4. *An Eulerian quasigroup of order n has column parity equal to*

$$\begin{cases} 0 & \text{if } n \equiv 1 \text{ or } n \equiv 2 \mod 4, \\ 1 & \text{if } n \equiv 0 \text{ or } n \equiv 3 \mod 4. \end{cases}$$

Proof. Suppose that $\circ$ is any binary operation on a set Q of cardinality n. We can form a digraph $D_\circ$ that for each $a, b \in Q$ has a vertex $[a, b, a \circ b]$ and an arc from $\big[a, b, a \circ b\big]$ to $\big[b, a \circ b, b \circ (a \circ b)\big]$. If $\circ$ is right-cancellative, then $D_\circ$ consists of directed cycles, and, when $(Q, \circ)$ is an Eulerian quasigroup, $D_\circ$ consists of a single directed cycle.

Consider the effect of a local switch where we change the operation $\circ$ by replacing $a \circ c = d$ and $b \circ c = e$ with $a \circ c = e$ and $b \circ c = d$. If $[a, c, d]$ and $[b, c, e]$ are in the same cycle of $D_\circ$, that cycle is split into two by the switch. On the other hand, if $[a, c, d]$ and $[b, c, e]$ are in different cycles, then these two cycles are merged. No other cycles are affected. Thus, in both cases, our local switch changes the number of cycles in $D_\circ$ by one. The local switch also changes the column parity of $(Q, \circ)$ by 1, since it applies a single transposition to ϕ_c.

In the case when $\circ$ is defined by the rule $a \circ b = a$ we see that $D_\circ$ has n loops, one on each vertex $[a, a, a]$ for $a \in Q$. It also has $n(n-1)/2$ cycles of length two, one from $[a, b, a]$ to $[b, a, b]$ and back again, for $a \neq b$. Thus there are $n + n(n-1)/2$ cycles in total. By an appropriate sequence of local switches we can move from $(Q, \circ)$ to any Eulerian quasigroup. In doing so, we reduce the number of cycles by $n - 1 + n(n-1)/2$. The column parity is initially 0, since ϕ_y is the identity for all y in Q. Thus, when we reach the Eulerian quasigroup, the column parity will be congruent to $\frac{1}{2}(n-1)(n+2)$ modulo 2. The result follows. □

COROLLARY 1. *There is no Eulerian quasigroup isotopic to any group of even order.*

Proof. By Propositions 3 and 4 of Wanless (2004b), any quasigroup isotopic to a group of even order has the wrong column parity to be Eulerian. □

In particular, it is not possible to create an Eulerian quasigroup by permuting the symbols of the cyclic group as we did for odd orders. Nevertheless, Eulerian quasigroups of even order do exist. To construct them, we permute the symbols of a quasigroup of the correct column parity. Suppose that p is an odd prime, s and t are positive integers, and $n = 2^s p^t$. We define a quasigroup $(\mathbb{Z}_n, \star)$ of order n by putting $v = n/p$, $0 \star b = b + v$ and $v \star b = b$ if v divides b, and $a \star b = a + b$ otherwise. In Tables 3 and 4 we give a permutation for each order $n = 2q$ and $n = 4q$ where $q < 100$ is an odd prime power. Again, these are extended to $q < 1000$ in the Supplementary Material. The interpretation is analogous to Table 1, except that the permutation should be applied to the symbols in the operation table of $(\mathbb{Z}_n, \star)$.

For example, when $n = 6$ we first construct the quasigroup $(\mathbb{Z}_6, \star)$ shown on the left of Fig. 6. Applying the permutation $(0, 4)(1, 5)(2, 3)$ from Table 3 gives the quasigroup $(\mathbb{Z}_6, \odot)$ shown on the right of Fig. 6. This in turn gives the circular design in Fig. 7.

Table 3. *Permutations for order $n = 2q$ for odd prime powers $q < 100$*

n	permutation	n	permutation	n	permutation	n	permutation	n	permutation
6	$(0, 4)(1, 5)(2, 3)$	34	$\pi_{0,10}$	62	$\pi_{0,2}\pi_{30,14}$	106	$\pi_{0,2}\pi_{14,34}$	158	$\pi_{0,2}\pi_{10,33}$
10	$(0, 9)(2, 5)$	38	$\pi_{0,2}\pi_{5,2}$	74	$\pi_{0,2}\pi_{30,6}$	118	$\pi_{0,2}\pi_{16,2}$	162	$\pi_{0,2}\pi_{33,23}$
14	$\pi_{0,2}\pi_{8,2}$	46	$\pi_{0,2}\pi_{11,2}$	82	$\pi_{0,2}\pi_{8,9}$	122	$\pi_{0,2}\pi_{27,17}$	166	$\pi_{0,2}\pi_{14,71}$
18	$\pi_{3,2}\pi_{8,2}$	50	$\pi_{0,1}\pi_{6,15}$	86	$\pi_{0,3}$	134	$\pi_{0,2}\pi_{13,10}$	178	$\pi_{0,2}\pi_{103,12}$
22	$\pi_{0,2}\pi_{12,1}$	54	$\pi_{0,4}\pi_{13,4}$	94	$\pi_{0,2}\pi_{53,13}$	142	$\pi_{0,2}\pi_{39,6}$	194	$\pi_{0,2}\pi_{18,76}$
26	$\pi_{0,2}\pi_{12,6}$	58	$\pi_{0,3}\pi_{10,10}$	98	$\pi_{0,8}\pi_{22,12}$	146	$\pi_{0,2}\pi_{6,60}$		

Table 4. *Permutations for order $n = 4q$ for odd prime powers $q < 100$*

n	permutation	n	permutation	n	permutation	n	permutation	n	permutation
12	$\pi_{0,1}\pi_{3,4}$	68	$\pi_{0,1}\pi_{3,8}$	124	$\pi_{0,3}\pi_{17,27}$	212	$\pi_{0,3}\pi_{21,17}$	316	$\pi_{0,3}\pi_{61,116}$
20	$\pi_{0,3}\pi_{7,1}$	76	$\pi_{0,3}\pi_{49,13}$	148	$\pi_{0,3}\pi_{67,3}$	236	$\pi_{0,3}\pi_{35,4}$	324	$\pi_{0,2}\pi_{11,109}$
28	$\pi_{0,1}\pi_{4,4}$	92	$\pi_{0,4}\pi_{9,10}$	164	$\pi_{0,3}\pi_{79,29}$	244	$\pi_{0,3}\pi_{9,28}$	332	$\pi_{0,3}\pi_{45,123}$
36	$\pi_{0,1}\pi_{9,12}$	100	$\pi_{0,11}\pi_{29,30}$	172	$\pi_{0,3}\pi_{137,6}$	268	$\pi_{0,3}\pi_{17,110}$	356	$\pi_{0,3}\pi_{50,35}$
44	$\pi_{0,3}\pi_{23,7}$	108	$\pi_{0,8}\pi_{17,31}$	188	$\pi_{0,3}\pi_{41,37}$	284	$\pi_{0,3}\pi_{23,86}$	388	$\pi_{0,3}\pi_{22,67}$
52	$\pi_{0,3}\pi_{11,15}$	116	$\pi_{0,3}\pi_{41,16}$	196	$\pi_{0,4}\pi_{11,53}$	292	$\pi_{0,4}\pi_{17,137}$		

$\star$	0	1	2	3	4	5
0	2	1	4	3	0	5
1	1	2	3	4	5	0
2	0	3	2	5	4	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\odot$	0	1	2	3	4	5
0	3	5	0	2	4	1
1	5	3	2	0	1	4
2	4	2	3	1	0	5
3	2	0	1	4	5	3
4	0	1	4	5	3	2
5	1	4	5	3	2	0

Fig. 6. Two quasigroups used in the construction of a circular design for six treatments.

$$(0, 0, 3, 2, 1, 2, 2, 3, 1, 0, 5, 1, 4, 1, 1, 3, 0, 2, 0, 4, 4, 3, 5, 3, 3, 4, 5, 2, 5, 5, 0, 1, 5, 4, 2, 4)$$

Fig. 7. Circular design for six treatments in 36 plots: it is balanced for neighbours at distances one and two.

Table 5. *Permutations for powers of two and three times a power of two*

n	permutation	n	permutation	n	permutation	n	permutation
8	$(0, 1)(2, 4)(3, 7)$	128	$\pi_{0,3}\pi_{15,14}$	24	$\pi_{0,1}\pi_{3,8}$	384	$\pi_{0,5}\pi_{30,76}$
16	$\pi_{0,4}$	256	$\pi_{0,1}\pi_{4,48}$	48	$\pi_{0,2}\pi_{8,4}$	768	$\pi_{0,3}\pi_{94,60}$
32	$\pi_{0,1}\pi_{7,2}$	512	$\pi_{0,3}\pi_{41,101}$	96	$\pi_{0,1}\pi_{8,12}$	1536	$\pi_{0,3}\pi_{317,158}$
64	$\pi_{0,3}\pi_{14,3}$			192	$\pi_{0,1}\pi_{12,27}$		

It remains to treat the case when $n = q$ or $n = 3q$ where q is a power of 2 in the range $8 \leqslant q \leqslant 1000$. Put $w = n/8$, and define the quasigroup $(\mathbb{Z}_n, \diamond)$ by $0 \diamond w = 7w \diamond 0 = 0$, $0 \diamond 4w = 5w \diamond w = w$, $0 \diamond 0 = 2w \diamond 2w = 2w$, $0 \diamond 2w = 2w \diamond 4w = 4w$, $2w \diamond 0 = 5w \diamond 4w = 6w \diamond w = 6w$, $6w \diamond 0 = 7w \diamond w = 7w$ and $a \diamond b = a + b$ otherwise. Applying the relevant permutation from Table 5 to the symbols in the operation table of $(\mathbb{Z}_n, \diamond)$ produces an Eulerian quasigroup. Together with Proposition 3, the examples we have found prove Theorem 3.

8. Further work

The most obvious outstanding questions are the proof or refutation of Conjecture 1, and the optimality of the semi-Eulerian sequences under model (2), but there are others.

In order to find neighbour-balanced designs in as small a number of plots as possible, we have specified that certain configurations should each appear exactly once. This could be generalized to r times for any positive integer r. When there is no solution for $r=1$, there might be solutions for some larger values of r. In particular, when $r=2$ then (0, 0, 0, 1, 1, 1, 0, 0) is a solution to the original problem for $n=2$, as are (0, 0, 1, 1, 2, 2, 0, 0, 2, 0, 2, 2, 1, 2, 1, 1, 0, 1) for $n=3$ and (1, 0, 3, 3, 3, 1, 2, 1, 1, 0, 0, 1, 3, 1, 3, 2, 0, 1, 1, 2, 3, 0, 0, 2, 0, 3, 0, 2, 2, 2, 3, 2) for $n=4$.

Theorem 5. *If there is a circular sequence for n treatments in which each ordered pair occurs exactly r times at distances one and two, then there is also a circular sequence in which each ordered pair occurs exactly $r+1$ times at distances one and two. The same is true when self-pairs are banned at distances one and two.*

Proof. Let Q be the set of n treatments. Build a digraph D_1 with vertices $[a, b]$ for a, b in Q. For each subsequence $[a, b, c]$ of the first sequence, put a directed edge from $[a, b]$ to $[b, c]$. Then D_1 is connected. Let $(Q, \circ)$ be any quasigroup on Q. For each vertex $[a, b]$, insert a new edge from $[a, b]$ to $[b, a \circ b]$, making a new digraph D_2. Then D_2 is connected, and the in-degree and out-degree of each vertex are both equal to $r+1$. Hence D_2 has an Eulerian trail, which gives a circular sequence with the desired property.

When self-neighbours are banned, omit vertices of the form $[a, a]$ and insist that $(Q, \circ)$ be idempotent. □

Alternatively, we might look for smaller designs. Equality of all variances of differences needs the factors for direct effects, left-neighbour effects, and right-neighbour effects to have the overall balance of Preece (1976). For a design with nk inner plots, a necessary condition is the existence of a balanced incomplete-block design for n treatments in n blocks of size k. Existence of a set of nk triples satisfying the conditions is the analogue of the existence of a quasigroup of order n. Whether there exist such sets that also have the Eulerian property seems to be a hard question.

The design at the end of § 5 has neighbour balance at distances one, two, and three. It would be suitable for a cross-over trial in which carry-over effects are expected on the three succeeding periods. Have such effects ever been recorded? Are there other such designs with only $n(n-1)$ inner plots?

The designs given by Dyke & Shelley (1976) have the further property that the circle or line can be divided into blocks of n consecutive plots in which each treatment occurs once. Then block effects can be fitted to allow for variation in space or time. Can this be done for any of the designs discussed in this paper?

Neighbour-balanced designs always present a problem for randomization. To avoid bias in the estimation of treatment effects, the actual treatments should be randomized to the treatment labels in the combinatorial design. In addition, the design can be rotated through a random number of places, and reversed with probability one half. However, this is not enough to make the randomization valid in the sense of Yates (1933). Bailey (1984) showed that, for some classes of neighbour-balanced design, there are some values of the parameters for which there exists a set of designs among which random choice leads to validity, while there are other parameter values for which no such set exists. Do such sets of designs exist for any of the types of neighbour-balanced design considered in this paper?

Within each of our three categories, all designs satisfying the combinatorial conditions are equally good if the assumed model is correct. If it is not, are some designs better than others for protecting against model inadequacy?

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Supplementary material

Supplementary material available at *Biometrika* online includes full versions of Tables 1–4 for $q < 1000$.

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