# Fast generation of some classes of planar graphs

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**Abstract**

In the talk we present an efficient algorithm to construct all non isomorphic triangulations of the sphere and their duals, that is: cubic polyhedra. Furthermore we present some extensions to the algorithm that make it applicable also for polyhedra, connected cubic planar graphs and triangulations of the disc.

## Introduction

Constructing complete lists of mathematical objects has a long tradition in mathematics. The 5 Platonic solids – that is the complete list of regular polyhedra – have already been determined by Theaetetus of Athens around 400 B.C. and play the central role in the 13th book of Euclid's Elements (around 300 B.C.).

When computers started to be used for enumeration purposes, again a class of polyhedra was one of the first classes to be generated: Already in 1965 D.W. Grace [8] listed all cubic polyhedra with up to 11 faces. After that for various classes of polyhedra enumeration methods have been proposed and lists have been constructed, see e.g. [3],[1], [11],[5].

Since some chemical molecules exhibit a polyhedral structure, the most famous among them probably being the fullerene $C_{60}$ ([9]), polyhedra and enumeration methods for polyhedra have not only been investigated by mathematicians,

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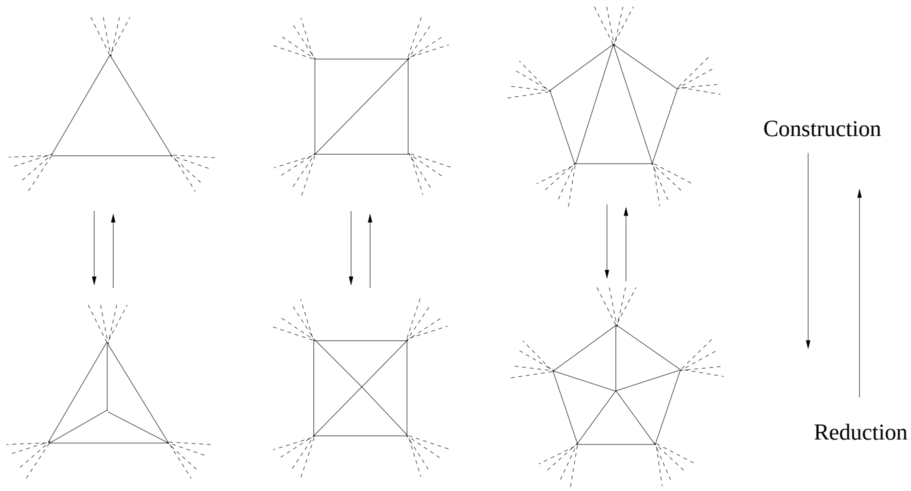


Fig. 1. The construction and reduction operations

but also by chemists. Lately, polyhedral, fullerene like structures have even been observed in astronomy (see [13]).

A survey on results on some classes of polyhedra can be found in the already classical book by Grünbaum [7] or in the book by Cromwell [4].

We will describe methods to generate the following classes:

**Triangulations:** This is the central part of the algorithm. We construct complete lists of pairwise non-isomorphic triangulations of the plane. The recursive construction method we used for this task was already given by Eberhard, Steinitz and Rademacher in [6] and [12]. It is depicted in figure 1.

**Cubic Polyhedra:** Dualizing the lists of triangulations we obtain complete lists of pairwise non-isomorphic cubic polyhedra.

**Polyhedra:** Since every planar graph is subgraph of a triangulation with the same number of vertices and since the connectivity of the graph is monotonically decreasing when edges are removed, all polyhedra – that is: all 3-connected simple planar graphs – can be obtained by deleting edges from triangulations. We use this strategy to construct complete lists of polyhedra. After every deletion, we have to check whether the graph is still 3-connected. For this check we do not have to regard the whole graph, but just an environment of the edge that was deleted.

**Triangulations of the disc:** Given a triangulation of the disc with $n$ vertices, placing the disc in the plane, inserting an additional vertex in the outer region and connecting it with all vertices on the boundary of the disc, we obtain a triangulation of the plane with $n+1$ vertices. The inverse operation allows us to construct triangulations of the disc from triangulations of the sphere.

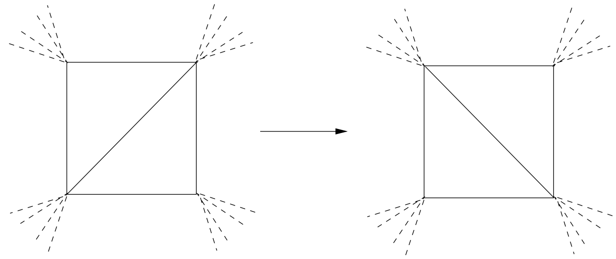

Fig. 2. The switch operation used to construct double edges and loops

**Cubic planar graphs:** Cubic planar graphs with 1- or 2-cuts cannot be obtained as duals of simple triangulations, but they can be obtained as duals of triangulations with double edges and/or loops, minimum degree 3 and no two faces sharing more than one edge. In order to obtain double edges and loops in our triangulations, we apply the switching operation depicted in figure 2 to our simple triangulations and dualize the resulting graphs. Note that the 4 vertices affected by this operation need not be distinct.

Of course efficient isomorphism rejection is an important part of the algorithm. We used orderly generation in the sense of [10] and the homomorphism principle (see e.g. [2]) for this task. Details how these methods can be applied in the different cases will be presented in the talk. The program implementing the above methods is fast enough to generate tens of thousands of graphs per second up to more than 100 000 graphs per second on a Linux Pentium II with 350 MHZ depending on the class.

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