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The Number of Matchings in Random Regular Graphs and Bipartite Graphs

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If $3 \leqslant k \leqslant (\log n)^{1/3}$ and n is even, then the number of 1-factors in random labelled k-regular graphs of order n has expectation

$$(\sqrt{2} + o(n^{-2/3}))\, e^{1/4}((k-1)^{k-1}/k^{k-2})^{n/2}$$

and variance

$$(1 + o(n^{-1/2}))\, e^{1/2}((k-1)^{k-1}/k^{k-2})^{n} \left(e^{-(2k-1)/4(k-1)^2} \left(\frac{k-1}{k-2} \right)^{1/2} - 1 \right).$$

A similar result is proved for random regular bipartite graphs.

The van der Waerden conjecture, proved by Egorychev [3] and Falikman [5], states that the permanent of a doubly stochastic n by n matrix is at least $n!/n^n$. Consequently if Λ_n^k is the set of n by n matrices with non-negative integer entries in which each row and column sum equals k then the permanent of a matrix in Λ_n^k is at least $k^n n!/n^n$. For fixed k and large n this bound is rather crude: there is no matrix in Λ_n^k whose permanent is $k^n n!/n^n$. However, it was proved recently by Schrijver and Valiant [13] that some matrix in Λ_n^k has permanent at most $k^{2n}/\binom{kn}{n}$. (See Schrijver [11] for related results.) Equivalently, there is an n by n bipartite multigraph which is regular of degree k and has at most $k^{2n}/\binom{kn}{n}$ complete matchings (1-factors). (In fact, Schrijver and Valiant showed that $k^{2n}/\binom{kn}{n}$ is the average number of 1-factors.)

Our main aim in this paper is to determine the asymptotic values of the first two moments of the number of 1-factors in random regular graphs and

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bipartite graphs. The emphasis of the paper is on random regular graphs but the results concerning bipartite graphs extend those announced by O'Neil [10], which, in turn, extended results of Erdős and Kaplansky [4]. A rather weak consequence of our results (already implied by those of O'Neil) is that for a fixed $k \geqslant 3$ there is an n by n bipartite graph (not multigraph) containing at most $(1+o(1))\, e^{-1/2} k^{2n}/\binom{kn}{n}$ 1-factors.

The systematic study of random regular graphs was started only recently, when Bender and Canfield [1] determined the asymptotic number of k-regular graphs. In [2] Bollobás gave a probabilistic proof of the formula, based on a simple model of the probability space of k-regular graphs. We start with an easy consequence of this model.

THEOREM 1. *For a fixed $\varepsilon > 0$ there is a function $\delta(n) = o(e^{(\log n)^{4/5}}/n)$ with the following property. Suppose $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$ are natural numbers, $\sum_{i=1}^{n} d_i = 2m$ is even,*

$$\Delta = d_1 \leqslant (\log n)^{1/3} \qquad \textit{and} \qquad m \geqslant \max\{\varepsilon \Delta n, (1+\varepsilon)\, n\}.$$

Suppose furthermore that F is a graph of maximum degree at most $(\log n)^{1/3}$, having vertex set $V = \{x_1, x_2, ..., x_n\}$. Denote by $\mathscr{L}(\mathbf{d}; F)$ the set of graphs with vertex set V whose degree sequence $(d(x_i))_1^n$ is exactly $\mathbf{d} = (d_i)_1^n$, and which do not share an edge with F. Then

$$|\mathscr{L}(\mathbf{d}; F)| = (1+\eta)\, e^{-\lambda - \lambda^2 - \mu} (2m)_m \Big/ \left\{ 2^m \prod_{i=1}^{n} d_i! \right\},$$

where

$$\lambda = \frac{1}{2m} \sum_{1}^{n} \binom{d_i}{2}, \qquad \mu = \frac{1}{2m} \sum_{x_i x_j \in E(F)} d_i d_j$$

and

$$|\eta| \leqslant \delta(n) = o(e^{(\log n)^{4/5}}/n).$$

Proof. In fact, considerably more is true than states in the theorem above. Let $W = \bigcup_{j=1}^{n} W_j$ be a fixed set of $2m = \sum_1^n d_j$ labelled vertices, where $|W_j| = d_j$. A partition of W into pairs will be called a *configuration.* The pairs of a configuration H are called the *edges* of H and the set of all configurations is denoted by Φ. Clearly,

$$|\Phi| = \frac{(2m)!}{2^m m!}. \tag{1}$$

An edge xy of a configuration is called a *loop* if x and y belong to the same set W_i and it is said to be a *forbidden edge* if $x \in W_i$, $y \in W_j$, and $x_i x_j$ is an edge of F. Finally, a *pair of edges* is said to be a *coupling* if they join the same two sets W_i and W_j. For a configuration $H \in \Phi$ denote by $X_1(H)$ the number of loops, by $X_2(H)$ the number of couplings and by $Y(H)$ the number of forbidden edges of H.

Turn Φ into a probability space by giving each configuration $H \in \Phi$ the same probability. Then X_1, X_2, and Y become random variables. As in [2], it is easily shown that under the conditions of our theorem the random variables X_1, X_2, and Y are asymptotically independent Poisson random variables with means λ, λ^2, and μ, with a certain bound on the speed of convergence. In particular, denoting by Φ_0 the set of configurations H with $X_1(H) = X_2(H) = Y(H) = 0$, we find that

$$|\Phi_0| = (1+\eta)\, e^{-\lambda-\lambda^2-\mu}\, |\Phi|, \tag{2}$$

where $|\eta| \leqslant \delta(n)$ and $\delta(n) = o(e^{(\log n)^{4/5}}/n)$ is some fixed function.

Given $H \in \Phi_0$ let $\phi(H)$ be the graph with vertex set V in which $x_i x_j$ is an edge iff some edge of H joins W_i to W_j. Clearly,

$$\mathscr{L}(\mathbf{d}; F) = \{\phi(H) \colon H \in \Phi_0\}$$

and for every $G \in \mathscr{L}(\mathbf{d}; F)$,

$$|\phi^{-1}(G)| = \prod_{i=1}^{n} d_i!,$$

so

$$|\mathscr{L}(\mathbf{d}; F)| = |\Phi_0| \Big/ \prod_{i=1}^{n} d_i!. \tag{3}$$

Putting (1), (2), and (3) together we arrive at the assertion of the theorem. ∎

Denote by $\mathscr{L}_{k\text{-reg}}(n)$ the set of k-regular graphs with vertex set $\{x_1, x_2, \dots, x_n\}$. Consider $\mathscr{L}_{k\text{-reg}}(n)$ as a probability space in which all points have the same probability. Denote by $X(G)$ the number of 1-factors of a graph G. Then $X = X(G)$ is a random variable on $\mathscr{L}_{k\text{-reg}}(n)$. Our main aim is to determine the asymptotic values of the first and second moments of X. The mean of X is easily approximated.

THEOREM 2. *Suppose $3 \leqslant k = k(n) \leqslant (\log n)^{1/3}$ and n is even. Then for some function $\delta(n) = o(e^{(\log n)^{4/5}}/n)$,*

$$\begin{aligned} E(X) &= (1+\eta_1)\, e^{1/4} \frac{n!}{(n/2)!} \frac{(kn/2)!}{(kn)!} \frac{((k-1)\,n)!}{((k-1)\,n/2)!} k^n \\ &= (1+\eta_2) \sqrt{2} e^{1/4} ((k-1)^{k-1}/k^{k-2})^{n/2}, \end{aligned}$$

where $|\eta_1| + |\eta_2| \leqslant \delta(n)$.

Proof. There are $F_n = n!/\{2^{n/2}(n/2)!\}$ 1-factors in the complete graph with vertex set V. By Theorem 1 the number of $(k-1)$-regular graphs on V containing no edge of a given 1-factor is

$$N_F = (1+\eta)\exp\left\{-\frac{k-2}{2}-\left(\frac{k-2}{2}\right)^2-\frac{(k-1)^2}{2(k-1)}\right\}$$
$$\times\frac{((k-1)\,n)!}{2^{(k-1)n/2}((k-1)\,n/2)!\,((k-1)!)^n}$$

while the number of k-regular graphs with vertex set V is

$$N = (1+\eta')\exp\left\{-\frac{k-1}{2}-\left(\frac{k-1}{2}\right)^2\right\}\frac{(kn)!}{2^{kn/2}(kn/2)!\,(k!)^n},$$

where $|\eta| + |\eta'| \leqslant \frac{1}{2}\delta(n)$ for some function $\delta(n) = o(e^{(\log n)^{4/5}}/n)$. By definition

$$E(X) = F_n N_F/N. \quad \blacksquare$$

THEOREM 3. *Suppose* $3 \leqslant k = k(n) \leqslant (\log n)^{1/3}$ *and n is even. Then*

$$\sigma^2(X)/E(X)^2 \leqslant (1+o(n^{-1/2}))\left\{e^{-(2k-1)/4(k-1)^2}\left(\frac{k-1}{k-2}\right)^{1/2}-1\right\}.$$

Proof. In order to make the calculations slightly easier to follow, we shall show that

$$E(X^2)/E(X)^2 \sim e^{-(2k-1)/4(k-1)^2}\left(\frac{k-1}{k-2}\right)^{1/2}.$$

It will be clear that our approximations are precise enough to yield the assertion of the theorem.

Let s be an even natural number. As before, denote by F_s the number of 1-factors on a set $W = \{y_1, y_2, ..., y_s\}$. Furthermore, denote by G_s the number of 1-factors on W that are disjoint from a given 1-factor. By the inclusion–exclusion principle

$$G_s = \sum_{i=0}^{s/2} (-1)^i \binom{s/2}{i} F_{s-2i},$$

where F_0 is defined to be 1. It is easily seen that as $s \to \infty$,

$$G_s \sim e^{-1/2} F_s.$$

Consider two 1-factors on $V = \{x_1, x_2, ..., x_n\}$, having exactly $n/2 - l$ edges in common. Set $\alpha = 2l/n$. Note that a k-regular graph containing these 1-factors has $(kn-n-2l)/2=(k-1-\alpha)\,n/2$ edges not belonging to these 1-factors. By Theorem 1 the number of such graphs is asymptotic to

$$T_l=e^{-\lambda_l-\lambda_l^2-\mu_l}((k-1-\alpha)\,n)!/\{((k-1-\alpha)\,n/2)!\,2^{(k-1-\alpha)n/2} \times ((k-1)!)^{(1-\alpha)n}\,((k-2)!)^{\alpha n}\}$$

where

$$\lambda_l=\frac{1}{(k-1-\alpha)\,n}\left\{(n-2l)\binom{k-1}{2}+2l\binom{k-2}{2}\right\} =\frac{(k-2)(k-1-2\alpha)}{2(k-1-\alpha)}$$

and

$$\mu_l=\frac{1}{(k-1-\alpha)\,n}\left\{\frac{n-2l}{2}(k-1)^2+2l(k-2)^2\right\} =\frac{(k-1)^2+\alpha(k^2-6k+7)}{2(k-1-\alpha)}.$$

Now we can write down an asymptotic formula for the expected number of ordered pairs of 1-factors in a k-regular graph:

$$E(X(X-1)\sim F_n\sum_{l=1}^{n/2}\binom{n/2}{l}G_{2l}T_l/N.$$

Consequently,

$$E(X^2)/E(X)^2\sim E(X(X-1))/E(X)^2\sim\sum_{l=1}^{n/2}N\binom{n/2}{l}G_{2l}T_l\Big/\{F_nN_F^2\}.$$

It is easily checked that the sum of terms belonging to small values of l is $o(n^{-1})$, say

$$\begin{aligned}E(X^2)/E(X)^2&\sim\sum_{l\geqslant n^{1/2}}^{n/2}N\binom{n/2}{l}G_{2l}T_l\Big/\{F_nN_F^2\}\\&\sim\sum_{l\geqslant n^{1/2}}^{n/2}e^{-1/2}N\binom{n/2}{l}F_{2l}T_l\Big/\{F_nN_F^2\}\\&=\sum_{l\geqslant n^{1/2}}^{n/2}R_l.\end{aligned}$$

In the range above we can use Stirling's formula $(a!\sim\sqrt{2\pi a}\,(a/e)^a)$ to obtain a uniform approximation of the factors. Note first that the exponent of e in the lth summand is

$$\begin{aligned}\varepsilon_l &= -1/2-(k-1)/2-(k-1)^2/4-\lambda_l-\lambda_l^2-\mu_l\\ &\quad +k-2+(k-2)^2/2+k-1\\ &= o(k).\end{aligned}$$

It is easily seen that if l is changed by $o(n/k^2)$ to l' (so that α changes by $o(k^{-2})$) then

$$\varepsilon_l = \varepsilon_{l'} + o(1).$$

By the Stirling approximation the product of the remaining parts of the factors of the lth summand has the following uniform approximation:

$$\begin{aligned}&(kn)!\,(n/2)!\,(2l)!\,((k-1-\alpha)\,n)!\,2^{n/2}(n/2)!\\ &\quad\times 2^{(k-1)n}(((k-1)\,n/2)!)^2\,(k!)^{2n}\\ &\div 2^{kn/2}(kn/2)!\,(k!)^n\,l!\,(n/2-l)!\,2^l l!\,((k-1-\alpha)\,n/2)!\\ &\quad\times 2^{(k-1-\alpha)n/2}((k-1)!)^{(1-\alpha)n}((k-2)!)^{\alpha n}\,n!\,(((k-1)\,n)!)^2\\ &\sim(\pi\alpha(1-\alpha)\,n)^{-1/2}\left(\frac{k(k-1-\alpha)}{(k-1)^2}\right)^{(k-1-\alpha)n/2}(k(1-\alpha))^{-(1-\alpha)n/2},\end{aligned}$$

where, as before, $\alpha=2l/n$. Set $\alpha_0=1-1/k$, $l_0=\alpha_0 n/2$, $l=l_0+(n/k)^{1/2}x$, $\alpha=2l/n=\alpha_0+(2/\sqrt{kn})\,x$. Then

$$\begin{aligned}S_l &= \left(\frac{k(k-1-\alpha)}{(k-1)^2}\right)^{k-1-\alpha}(k(1-\alpha))^{\alpha-1}\\ &= (1-2k^{1/2}x/((k-1)^2 n^{1/2}))^{(k-1)^2/k-2x/\sqrt{kn}}\\ &\quad\times\left(1-2\left(\frac{k}{n}\right)^{1/2}x\right)^{2x/\sqrt{kn}-1/k}.\end{aligned}$$

It is easily seen that the maximum of S_l is 1, attained at $x=0$. (Strictly speaking, this is not realized if $\alpha_0 n/2$ is not an integer.) Furthermore, if $|x|>n^{1/7}$ then for sufficiently large n we have

$$S_l^{n/2}\leqslant n^{-2},$$

so

$$\sum_{l\geqslant n^{1/2}}^{n/2} R_l\sim e^{-\varepsilon_{l_0}}\sum{}'\,(\pi\alpha_0(1-\alpha_0)\,n)^{-1/2}\,S_l^{n/2},$$

where the summation is over the values of l satisfying $|x| \leqslant n^{1/7}$, that is, $|l-l_0| \leqslant n^{9/14}/k^{1/2}$.

Straightforward calculations give that in the range $|x| \leqslant n^{1/7}$ we have uniformly

$$\begin{aligned} S_l^{n/2} &\sim \exp\{-((kn)^{1/2}\, x/(k-1)^2 + kx^2/((k-1^4)) \\ &\quad \times ((k-1)^2/k - 2x/(kn)^{1/2}) - ((kn)^{1/2}\, x + kx^2)(2x/(kn)^{1/2} - 1/k)\} \\ &\sim \exp\{-x^2(1-(k-1)^{-2})\}. \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_{l \geqslant n^{1/2}}^{n/2} R_l &\sim e^{-\varepsilon_{l_0}}(\pi\alpha_0(1-\alpha_0)\, n)^{-1/2}\, (n/k)^{1/2} \\ &\quad \times \int_{-\infty}^{\infty} \exp(-x^2(1-(k-1)^{-2})\, dx \\ &= e^{-\varepsilon_{l_0}}(\pi(k-1)\, n/k^2)^{-1/2}\, (n/k)^{1/2}\, (\pi(k-1)^2/k(k-2))^{1/2} \\ &= e^{-\varepsilon_{l_0}} \left(\frac{k-1}{k-2}\right)^{1/2}. \end{aligned}$$

Finally,

$$\lambda_{l_0} = \frac{(k-2)^2}{2(k-1)} \qquad \text{and} \qquad \mu_{l_0} = \frac{1}{2} + \frac{(k-2)^2}{k-1},$$

so after some calculations

$$\varepsilon_{l_0} = -(2k-1)/(4(k-1)^2).$$

Consequently, putting it all together,

$$E(X^2)/E(X)^2 \sim e^{-(2k-1)/4(k-1)^2} \left(\frac{k-1}{k-2}\right)^{1/2}. \quad \blacksquare$$

COROLLARY 4. *If $k = k(n) \leqslant (\log n)^{1/3}$ and $k(n) \to \infty$ and $\omega(n) \to \infty$, $\omega(n) = o(k^{1/2})$, then a.e. k-regular graph of order n contains at least*

$$\begin{aligned} &\left(1 - \frac{\omega(n)}{k^2(n)}\right) 2^{1/2} e^{1/4}((k-1)^{k-1}/k^{k-2})^{n/2} \\ &\quad = (1+o(1))\, 2^{1/2} e^{1/4}((k-1)^{k-1}/k^{k-2})^{n/2} \quad \textit{1-factors.} \end{aligned}$$

Proof. The assertion is immediate from Chebyshev's inequality since, by Theorem 3, $\sigma^2(X)/E(X)^2 = O(1/k^2)$. $\blacksquare$

Let us turn to bipartite graphs. First we need an analog of Theorem 1 for bipartite graphs.

THEOREM 5. *Suppose* $\varepsilon > 0$, $m = m(n) \leqslant n$, $1 \leqslant d_1 \leqslant d_2 \leqslant \cdots \leqslant d_m \leqslant (\log m)^{1/3}$, $1 \leqslant d'_1 \leqslant d'_2 \leqslant \cdots d'_n \leqslant (\log m)^{1/3}$ *and* $M = \sum_1^m d_i = \sum_1^n d'_i \geqslant \max\{(2+\varepsilon)\, n, \varepsilon d_m, \varepsilon d'_n\}$. *Suppose furthermore that F is a bipartite graph with classes* $V = \{x_1, x_2, ..., x_m\}$ *and* $W = \{y_1, y_2, ..., y_n\}$ *having maximum degree at most* $(\log m)^{1/3}$. *Then the number of bipartite graphs with vertex classes V and W, degree sequences* $d(x_i) = d_i$ *and* $d(y_j) = d'_j$, $i = 1, ..., m$, $j = 1, ..., n$, *and not containing any edge of F is*

$$(1+\eta)\, e^{-\lambda-\mu} M! \Big/ \left\{\prod_1^m d_i! \prod_1^n d_j!\right\},$$

where

$$\lambda = \frac{2}{M^2} \left(\sum_1^m \binom{d_i}{2}\right) \left(\sum_1^n \binom{d'_j}{2}\right),$$

$$\mu = \frac{1}{M} \sum \{d_i d'_j : x_i y_j \in E(F)\},$$

and

$$\eta = o(n^{-3/4}).$$

The proof of this theorem is similar to that of Theorem 1. As there, we not only get the result claimed here, but also the convergence of the appropriate random variables. Once again we omit the details.

THEOREM 6. *Let* $3 \leqslant k = k(n) \leqslant (\log n)^{1/3}$ *and let* $\mathscr{B}(n, n; k\text{-reg})$ *be the probability space consisting of all labelled k-regular n by n bipartite graphs. As before, for* $G \in \mathscr{B}(n, n; k\text{-reg})$ *denote by* $X = X(G)$ *the number of* 1*-factors of G. Then*

$$E(X) = (1 + o(n^{-3/4}))\, e^{-1/2} \left(\frac{2\pi(k-1)\, n}{k}\right)^{1/2} \left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n$$

and

$$\sigma^2(X)/E(X)^2 = (1 + o(n^{-1/2})) \left\{e^{-1/\{2(k-1)^2\}} \frac{k-1}{(k(k-2))^{1/2}} - 1\right\}.$$

Proof. Since there are $n!$ 1-factors from V to W, by Theorem 5

$$E(X) = (1 + o(n^{-3/4}))\, e^{-\lambda_0 - \mu_0} n! \, \frac{((k-1)\, n)!\, (k!)^{2n}}{((k-1)!)^{2n}\, (kn)!}\, e^{\lambda + \mu},$$

where λ_0 and μ_0 are the appropriate values in the $(k-1)$-regular case with F a 1-factor:

$$\lambda_0 = \frac{(k-2)^2}{2} \qquad \text{and} \qquad \mu_0 = k-1,$$

and λ, μ are the values in the k-regular case with F being empty:

$$\lambda = \frac{(k-1)^2}{2} \qquad \text{and} \qquad \mu = 0.$$

Hence,

$$E(X) = (1 + o(n^{-3/4}))\, e^{-1/2} n! \, \frac{((k-1)\,n)!}{((k-1)!)^{2n}} \frac{(k!)^{2n}}{(kn)!}.$$

By Stirling's formula this implies the first assertion.

In what follows we shall sketch a proof of the asymptotic formula for $E(X^2)/E(X)^2$.

Let F_1 and F_2 be two 1-factors from V to W sharing $n-l=(1-\alpha)\,n$ edges. What is the asymptotic number of graphs $G \in \mathscr{B}(n, n; k\text{-reg})$ containing both F_1 and F_2? By Theorem 5 it is asymptotically

$$S_l = e^{-\lambda_l - \mu_l}((k-1-\alpha)\,n)!/\{((k-1)!)^{2(1-\alpha)n}((k-2)!)^{2\alpha n}\},$$

where

$$\lambda_l = \frac{2}{(k-1-\alpha)^2}\left\{(1-\alpha)\binom{k-1}{2} + \alpha\binom{k-2}{2}\right\}^2$$

and

$$\mu_l = \frac{1}{k-1-\alpha}\{(1-\alpha)(k-1)^2 + 2\alpha(k-2)^2\}.$$

Note that if $n \to \infty$ then the number of ordered pairs of edge disjoint 1-factors from V to W is asymptotic to $e^{-1}(n!)^2$. Hence if $l \to \infty$ then the number of ordered pairs of 1-factors from V to W sharing exactly $n-l$ edges is asymptotic to

$$T_l = n! \binom{n}{l} e^{-1} l! = e^{-1}(n!)^2/(n-l)!.$$

Hence, writing N for the total number of graphs, we find that if $l \to \infty$ then the expected number of ordered pairs of 1-factors in $G \in \mathscr{B}(n, n; k\text{-reg})$ sharing $n-l$ edges is asymptotic to

$$S_l T_l / N.$$

Standard approximations show that

$$E(X^2) \sim \sum{}' S_l T_l / N,$$

where $\sum'$ denotes summation over the values of l satisfying $|l-(1-1/k)\,n| < n^{1/2}(\log n)^2$. It is easily seen that in this range λ_l and μ_l are almost constant:

$$\lambda_l = \frac{(k-2)^4}{2(k-1)^2} + c_1 x/n + O(x^2/n^2)$$

and

$$\mu_l = \frac{2(k-2)^2}{k-1} + 1 + c_2 x/n + O(x^2/n^2),$$

where c_1 and c_2 are bounded functions of k and x is given by $l = (1-1/k)\,n + x((k-1)/k)(n/(k-2))^{1/2}$. Hence

$$-\lambda_l - \mu_l + \lambda = -\frac{1}{2(k-1)^2} + cx/n + O(x^2/n^2),$$

where c is a bounded function of k.

Using this approximation we find that

$$\begin{aligned}
&e^{1/\{2(k-1)^2\}} E(X^2)/E(X)^2 \\
&\sim \sum{}' \frac{(n!)^2}{((1-\alpha)\,n)!} \frac{((k-1-\alpha)\,n)!}{((k-1)!)^{2(1-\alpha)n}((k-2)!)^{2\alpha n}} \\
&\quad \times \frac{((k-1)!)^{4n}}{(n!)^2\,(((k-1)\,n)!)^2} \frac{((kn)!)^2}{(k!)^{4n}} \frac{(k!)^{2n}}{(kn)!} \\
&\sim \sum{}' \left(\frac{(k-1-\alpha)\,k}{(1-\alpha)(k-1)^2\, 2\pi n} \right)^{1/2} k^{(k-2)n} (k-1)^{-2(k-1-\alpha)n} \\
&\quad \times (k-1-\alpha)^{k-1-\alpha} (1-\alpha)^{(\alpha-1)n} \\
&\sim \left(\frac{k}{2\pi n} \right)^{1/2} \sum{}' \left\{ (k(1-\alpha))^{\alpha-1} \left(\frac{k(k-1-\alpha)}{(k-1)^2} \right)^{(k-1-\alpha)} \right\}^n,
\end{aligned}$$

where $\alpha = \alpha(l) = l/n$ and $\sum'$ is as before.

Standard calculations yield that the right-hand side above is asymptotic to

$$\left(\frac{k}{2\pi n} \right)^{1/2} \frac{k-1}{k} \left(\frac{n}{k-2} \right)^{1/2} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \frac{k-1}{(k(k-2))^{1/2}}.$$

This gives

$$E(X^2)/E(X)^2 \sim e^{-1/\{2(k-1)^2\}} \frac{k-1}{(k(k-2))^{1/2}}. \quad \blacksquare$$

COROLLARY 7. *If $k = k(n) \leqslant (\log n)^{1/3}$ and $k \to \infty$ then a.e. $G \in \mathscr{B}(n, n;$ k-reg$)$ contains at least*

$$\left(1 - \frac{1}{3k}\right)\left(\frac{2\pi}{e}\right)^{1/2} n^{1/2}\{(k-1)^{k-1}/k^{k-2}\}^n$$

1-factors.

Proof. The assertion is immediate by Chebyshev's inequality, since $\sigma^2(X)/E(X)^2 = O(1/k^3)$. $\blacksquare$

In conclusion let us note a result concerning matchings in bipartite graphs with unequal classes. Let $1 \leqslant r < s$ be fixed integers. Consider the set of bipartite graphs with vertex classes U and W such that $|U| = rn$, $|W| = sn$, every vertex in U has degree s and every vertex in W has degree r. The expected number of matchings from U into W is asymptotic to

$$\exp\left(-\frac{r-1}{2s-1}\right)\left(\frac{s-1}{s-r}\right)^{1/2}\left(\frac{(s-1)^{r(s-1)}}{s^{rs-s-r}(s-r)^{s-r}}\right)^n.$$

Late Notes. The results of this paper can be established over a wider range of degrees by using [8 and 9] in place of Theorems 1 and 5. In particular, this establishes Theorems 2, 3, and 6, with different error terms, for $k = o(n^{1/3})$. Some results of this nature can be also established for random regular graphs of very high degree. For example, Theorem 3.5 of [7] applied to Theorem 6.2 of [6] yields the following.

THEOREM 8. *Suppose $0 \leqslant l = l(n) = O(n^{1-\varepsilon})$ for some $\varepsilon > 0$. Then the average number of 1-factors in a random regular bipartite graph of degree $n - l$ and order $2n$ is uniformly*

$$n!(1 - l/n)^n\left(1 + \frac{l}{2n} + \frac{l(15l-4)}{24n^2} + \frac{l(37l^2-4l-12)}{48n^3}\right.$$
$$\left. + \frac{5415l^4 + 360l^3 - 2080l^2 - 2688l + 1440}{5760n^4} + O(l^5/n^5)\right).$$

Finally, we can obtain the average number of 1-factorizations (with labelled 1-factors) of regular biartite graphs by dividing the number of Latin rectangles by the number of graphs. Using [6 and 8], we obtain

THEOREM 9. *Let* $1 \leqslant k = k(n) = O(n^{1-\varepsilon})$ *for some* $\varepsilon > 0$. *Then the average number of* 1-*factorizations of random regular bipartite graphs of order* $2n$ *and degree* k *is uniformly*

$$\frac{(n!)^k \, (k!)^{2n}}{(nk)!} \exp\left(-\frac{k-1}{2} + O(k^3/n)\right).$$

Other results on this problem can be found in [11 and 12].

REFERENCES

1. E. A. BENDER AND E. R. CANFIELD, The asymptotic number of labelled graphs with given degree sequences, *J. Combin. Theory Ser. A* **24** (1978), 296–307.
2. B. BOLLOBÁS, A probabilistic proof of an asymptotic formula for the number of labelled regular graphs, *European J. Combin.* **1** (1980), 311–316.
3. G. P. EGORYCHEV, A solution of a problem of van der Waerden for permanent, preprint 1Fso–13M, Inst. Fiziki im. L. V. Kirenskogo, USSR Akad. Nauk, Siberian branch, Krasnoiarsk, 1980. [Russian]
4. P. ERDŐS AND I. KAPLANSKY, The asymptotic number of Latin rectangles, *Amer. J. Math.* **68** (1946), 230–236.
5. D. I. FALIKMAN, A proof of van der Waerden's conjecture on the permanent of a doubly stochastic matrix, *Mat. Zametki* **29** (1981), 931–938.
6. G. D. GODSIL AND B. D. MCKAY, Asymptotic enumeration of Latin rectangles, to appear.
7. B. D. MCKAY, Subgraphs of random graphs with specified degrees, *Congr. Numer.* **33** (1981), 213–223.
8. B. D. MCKAY, Asymptotics for 0–1 matrices with prescribed line sums, *in* "Enumeration and Design," pp. 225–238, Acadeic Press, New York/London, 1984.
9. B. D. MCKAY, Asymptotics for symmetric 0–1 matrices with prescribed row sums, *Ars Combin.* **19A** (1985), 15–26.
10. P. E. O'NEIL, Asymptotics and random matrices with row-sum and column-sum restrictions, *Bull. Amer. Math. Soc.* **75** (1969), 1276–1282.
11. A. SCHRIJVER, "On the Number of Edge Colourings of Regular Bipartite Graphs," A and E Report 18/80, University of Amsterdam, 1980.
12. A. SCHRIJVER, Bounds on permanents, and the number of 1-factors and 1-factorizations of biartite graphs, *in* "Surveys in Combinatorics," pp. 107–134, Cambridge Univ. Press, London/New York, 1983.
13. A. SCHRIJVER AND W. G. VALIANT, On lower bounds for permanents, *Indag. Math.* **42**, Fas. 4 (1980).