Asymptotic Properties of Labeled Connected Graphs

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ABSTRACT

We prove various properties of $C(n, q)$, the set of n-vertex q-edge labeled connected graphs. The domain of validity of the asymptotic formula of Erdős and Rényi for $|C(n, q)|$ is extended and the formula is seen to be the first term of an asymptotic expansion. The same is done for Wright's asymptotic formula. We study the number of edges in a random connected graph in the random edge model $\mathcal{G}_{n,p}$. For certain ranges of n and q, we determine the probability that a random edge (resp. vertex) of a random graph in $C(n, q)$ is a bridge (resp. cut vertex). We also study the degrees of random vertices.

1. INTRODUCTION

In this article graphs are labeled, and an (n, q) graph is one having n vertices and q edges. Let $C(n, q)$ be the set of connected (n, q) graphs and let $c(n, q) =$

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$|C(n, q)|$. We will often speak of things chosen at random. Unless stated otherwise, this means uniformly at random. For example, "Let e be a random edge of $C(n, q)$" means that we choose a graph $G \in C(n, q)$ using the uniform distribution and then an edge e of G using the uniform distribution. In [1] we obtained an asymptotic formula for $c(n, q)$ as $n, q \to \infty$. Here we apply this result to study various properties of these graphs. In particular, we study

(a) the asymptotics of $c(n, q)$ when q/n is near 1 and when q/n is large, developing expansions whose first terms were found by Wright [3] and Erdős and Rényi [2], respectively,
(b) connected graphs in the edge probability model for random graphs, determining their probability and the distribution of the number of edges;
(c) the expected number and nature of bridges in $C(n, q)$;
(d) the expected number and degree of cut and noncut vertices in $C(n, q)$.

Throughout we let

$$N = \binom{n}{2}, \qquad k = q - n \qquad \text{and} \qquad x = q/n = 1 + k/n\,. \tag{1.1}$$

The function $y = y(x) \geq 0$ is defined implicitly by

$$2xy = \log\left(\frac{1+y}{1-y}\right), \tag{1.2}$$

for $1 \leq x < \infty$. We will rely heavily on the results of [1], particularly the following.

Theorem 1.1. *With N, k, x, and y defined as above, the number of connected (n, q) graphs, $c(n, q)$ is given by*

$$c(n, q) = \binom{N}{q} w_k e^{n\varphi(x) + a(x)}(1 + O((k+1)^{1/16}/n^{9/50})) \tag{1.3}$$

as $n \to \infty$, uniformly for $0 \leq k \leq N - n$, where $\varphi(1) = 2/e$, $a(1) = 2 + \frac{1}{2}\log(3.2)$,

$$e^{\varphi(x)} = \frac{2e^{-x}y^{1-x}}{\sqrt{1-y^2}} = \frac{2e^{-x(1-y)}y^{1-x}}{1+y}, \tag{1.4}$$

$$a(x) = x(x+1)(1-y) + \log(1 - x + xy) - \frac{1}{2}\log(1 - x + xy^2) \tag{1.5}$$

and $w_k = (1 - 4/((k+1) + O((k+1)^{-2}))$. The exact value of w_k is given by [1, (1.6), (3.20)].

We also recall that

$$\exp(-\varphi'(x)) = y\,. \tag{1.6}$$

2. THE ERDŐS RÉNYI FORMULA

Recall the formula of Erdős and Rényi [2] for the number of labeled connected graphs with n vertices and q edges

$$c(n, q) \sim \binom{N}{q} \exp(e^{-2\mu})$$

as $n \to \infty$ with $q = \frac{1}{2} n \log n + \mu n + o(n)$. With $\epsilon = 1/4$ and $m = 2$, this formula is a consequence of the following theorem, which extends the Erdős–Rényi formula to lower values of q.

Theorem 2.1. *Let $\epsilon > 0$ and $m > 1/(2\epsilon)$ be fixed. Let the power series $f(u) = \sum_{t>0} f_t(x)u^t$ be defined implicitly by*

$$f(u) = u(1 - f(u))e^{4xf(u)} \tag{2.1}$$

and let

$$g(u) = \sum_{t=1}^{\infty} g_t(x)u^t = -2xf(u) - \log(1 - f(u)) - (x - 1)\log(1 - 2f(u)) . \tag{2.2}$$

Then $g_t(x)$ is a polynomial of degree $t - 1$ and

$$c(n, q) \sim \binom{N}{q} \exp\left\{n \sum_{t=1}^{m-1} g_t(x)e^{-2tx}\right\}$$

uniformly for $q > \epsilon n \log n$. The first few values of $g_t(x)$ are

$$g_1(x) = -1 , \qquad g_3(x) = -8x^2 - \frac{4}{3}x - \frac{1}{3} ,$$

$$g_2(x) = -2x - \frac{1}{2} , \quad g_4(x) = -\frac{128}{3}x^3 - 2x - \frac{1}{4} .$$

Proof. By [1, (3.8), (3.12)], $a(x) \to 0$ as $x \to \infty$. By the Lagrange inversion formula, $g_t(x)$ is the coefficient of f^{t-1} in

$$\frac{1}{t}\left(\frac{1}{1-f} + \frac{2(x-1)}{1-2f} - 2x\right)(1-f)^t e^{4txf} ,$$

which can be seen to be a polynomial in x of degree $t - 1$ with a bit of algebra. (In fact, for $t > 1$, the leading coefficient is $-4(4t)^{t-2}/t!$.) If p_1, p_2, and p_3 are polynomials in the variable f with positive coefficients, the coefficient of f^{t-1} in their product is bounded by $p_1(1)p_2(1)p_3(1)$. Thus,

$$\begin{aligned} |g_t(x)| &\leq \frac{1}{t}(t + 2(x-1)2^t + 2x)2^t(t(4xt)^{t-1}/(t-1)!) \\ &\leq 6(16e)^t x^t , \end{aligned}$$

using very crude bounds. Certainly, though, $g(u)$ converges for $0 \le u \le e^{-2x}$, and

$$|g^{(m)}(u)| \le K_m x^m , \qquad 0 \le u \le e^{-2x} , \tag{2.3}$$

where K_m is a constant depending only on m.

Let $\delta = (1-y)/2$. By (1.2) and some algebra,

$$\delta = e^{-2x}(1-\delta)e^{4x\delta} .$$

Comparing this with (2.1), we see that $\delta = f(e^{-2x})$. From (2.2) and the rightmost part of (1.4), $\varphi(x) = g(e^{-2x})$. To complete the proof, we note that by Taylor's theorem with remainder, (2.3), the choice of m, and the fact that $x \ge \epsilon \log n$,

$$g(e^{-2x}) - \sum_{t=1}^{m-1} g_t(x) e^{-2tx} = o(1/n) . \qquad \blacksquare$$

3. THE WRIGHT FORMULA

In [3] Wright proved that the number of connected sparsely edged graphs is given, for $k = o(n^{1/3})$, by

$$c(n, n+k) = d(3\pi)^{1/2}(e/12k)^{k/2} n^{n+\frac{1}{2}(3k-1)}(1 + O(k^{-1}) + O(k^{3/2}/n^{1/2})) ,$$

in which d is a constant which he evaluated to six decimal places. Later, Meertens proved that $d = 1/2\pi$, and his proof appears in [1]. Noting the value of d and that w_k, which is defined below, is equal to $1 + O(k^{-1})$, the next theorem shows, by taking $m = 2$ and $\epsilon > 1/2$, that Wright's formula is valid for an actually wider range of k than originally proved. Moreover, Wright's formula appears as the first term of an asymptotic expansion which allows larger k.

Theorem 3.1. *There exists a sequence C_t of constants such that for each fixed $\epsilon > 0$ and integer $m > 1/\epsilon$.*

$$c(n, q) = \sqrt{\frac{3}{\pi}} \frac{w_k}{2} \left(\frac{e}{12k}\right)^{k/2} n^{n+(3k-1)/2} \times \exp\left\{ \sum_{t=1}^{m-2} \frac{C_t k^{t+1}}{n^t} + O\left(\frac{k^m}{n^{m-1}} + \frac{k^{1/2}}{n^{1/2}} + \frac{(k+1)^{1/16}}{n^{9/50}} \right) \right\} \tag{3.1}$$

uniformly for $k = O(n^{1-\epsilon})$. The first few values of the constants are

$$C_1 = -\frac{1}{2}, \; C_2 = \frac{701}{2100}, \; C_3 = -\frac{263}{1050}, \; C_4 = \frac{538\,859}{2\,695\,000} .$$

Proof. Let $p(k/n)$ stand for a power series in k/n having nonzero radius of convergence and no constant term. The value of $p(k/n)$ is not necessarily the same at each occurrence. By (1.2) and standard complex analysis arguments, $y^2 = p(k/n)$ and the linear term of this power series is $3k/n$. Thus $\log(1-y^2) = p(k/n)$ and $\log(y\sqrt{n/3k}) = p(k/n)$. It follows that

$$(1-y^2)^{-n/2}y^{-k} = \left(\frac{3k}{ne^3}\right)^{-k/2} \exp\{kp(k/n)\}\,. \tag{3.2}$$

By Stirling's formula,

$$(n+k)! = \sqrt{2\pi n}\, n^{n+k} e^{-n}(1+O(k/n))\exp\{kp(k/n)\}\,. \tag{3.3}$$

By [1, (3.23), (3.25)],

$$(n+k)!\binom{N}{n+k} = (n^2/2)^{n+k} e^{-2}(1+O(k/n)) \tag{3.4}$$

and

$$a(1+k/n) = \frac{1}{2}\log(3/2) + 2 + O(k^{1/2}/n^{1/2})\,. \tag{3.5}$$

Noting that $(y^{1-x})^n = y^{-k}$ and using (1.4) and (3.2–5) in (1.3), we obtain the theorem. ∎

4. THE CONNECTEDNESS OF A RANDOM GRAPH

Let $R_p(n, q)$ be the probability that an n vertex graph constructed at random with edge probability $p = p(n)$ is connected and has q edges; that is,

$$R_p(n, q) = p^q(1-p)^{N-q}c(n, q)\,, \tag{4.1}$$

and let $\hat{R}_p(n, q)$ be the approximation of $R_p(n, q)$ suggested by (1.3); that is,

$$\hat{R}_p(n, q) = \left(1 - \frac{4}{k}\right)\binom{N}{q} p^q(1-p)^{N-q} \exp(n\varphi(x) + a(x))\,.$$

Note that although $R_p(n, q)$ is defined only for integral values of q, we define $\hat{R}_p(n, q)$ for nonintegral q by evaluating the binomial coefficient $\binom{N}{q}$ as $N!/\Gamma(q+1)\Gamma(N-q+1)$. We define a random variable $X_p(n)$ to be the number of edges of a random graph with edge probability p conditioned on the event that the graph is connected. Thus

$$\text{Prob}\{X_p(n) = q\} = \frac{R_p(n, q)}{\sum_q R_p(n, q)}\,. \tag{4.2}$$

In this section, we will determine the limiting behavior of $X_p(n)$ as $n \to \infty$.

Theorem 4.1. *Define*

$$y_0 = \tanh(pn/2)\,, \qquad q_0 = n^2p/2y_0\,, \qquad x_0 = q_0/n \tag{4.3}$$

and

$$\frac{1}{\sigma^2} = \frac{1}{q_0} - \frac{\varphi''(x_0)}{n}\,. \tag{4.4}$$

Then the following are true

(*a*) *Equation* (1.2) *holds for* x_0 *and* y_0.
(*b*) *If* $p^2n^3 \to \infty$ *and* $p = O(\log n/n)$, *then* $q_0 - n \to \infty$ *and* $(X_p(n) - q_0)/\sigma$ *converges to the normal distribution* $\mathcal{N}(0, 1)$ *as* $n \to \infty$.
(*c*) *If* $p = O(n^{-3/2})$ *as* $n \to \infty$, *then*

$$\operatorname{Prob}\{X_p(n) - n = k\} \sim \alpha_k = \begin{cases} (2\lambda/c)\sqrt{\pi/3}\,, & \text{for } k = -1\,; \\ \pi\lambda/\sqrt{6}\,, & \text{for } k = 0\,; \\ \lambda w_k(ce/12k)^{k/2}\,, & \text{otherwise}\,; \end{cases}$$

where $c = pn^{3/2}$, w_k *is given by* [1, (1.6)], *and* λ *is chosen so that* $\Sigma\, \alpha_k = 1$.
(*d*) *If* $p \geq C \log n/n$ *and* $C > 1$, *then* $X_p(n)$ *is approximately a binomial distribution with parameters* p *and* N; *more precisely*, $\operatorname{Prob}\{X_p(n) = q\} = \binom{N}{q} p^q(1-p)^{N-q} + O(n^{1-C})$ *uniformly in* n *and* q.

We will actually prove somewhat more than this because we will obtain big-oh estimates for the errors in the distributions. These estimates are given in Lemmas 4.2 and 4.3.

In order to establish the theorem, we will use the three lemmas. We first state the lemmas and then give the proofs.

Lemma 4.1. *The function* k_0, *equal by definition to* $q_0 - n$, *is an increasing function of* p *and*

$$k_0/2 < \sigma^2 < 4k_0\,. \tag{4.5}$$

When $0 < p = o(n^{-1})$,

$$y_0 = \frac{1}{2}\, pn - \frac{1}{24}\, p^3n^3 + O(p^5n^5)\,; \tag{4.6}$$

$$k_0 = q_o - n = \frac{1}{12}\, p^2n^3 + O(p^4n^5)\,; \tag{4.7}$$

$$\frac{1}{\sigma^2} = \frac{1}{2k_0} + O(1/n)\,. \tag{4.8}$$

Lemma 4.2. *When* $n^{-11/8} \leq p = O(\log n/n)$, $q_0 + r$ *is integral, and* $|r| \leq n^{1/32}\sigma$, *we have*

$$R_p(n, q_0 + r) = \hat{R}_p(n, q_0) \exp\{-r^2/2\sigma^2 + O(n^{-1/40})\}\,.$$

Further,

$$\sum_q R_p(n, q) = \sqrt{2\pi}\sigma \hat{R}_p(n, q_0)(1 + O(n^{-1/40})) .$$

Lemma 4.3. *Fix $0 < c < 1$. When $p^2n^3 \to \infty$, $p = O(n^{-11/8})$, $q_0 + r$ is integral, and $|r| \le ck_0$, we have*

$$R_p(n, q_0 + r) = \hat{R}_p(n, q_0) \exp\{-r^2/2\sigma^2 + O(|r|^3/k_0^2 + k_0^{3/2}/n^{1/2})\} .$$

Further, for any fixed $\delta > 0$

$$\sum_q R_p(n, q) = \sqrt{2\pi}\sigma \hat{R}_p(n, q_0)(1 + O(k_0^{\sigma - 1/2})) .$$

Proof (*of theorem from lemmas*). Result (a) follows from noting that (1.2) is equivalent to $y = \tanh(xy)$. The first part of result (b) follows from Lemma 4.1, first for small p from $k_0 \sim \frac{1}{12} p^2n^3$ and then for larger p by the monotonicity of k_0. The rest of (b) follows from Lemmas 4.2 and 4.3. Result (c) follows by simple calculation from (4.1), Cayley's formula n^{n-2} for the number of trees, and [1, (1.9)]. Finally, result (d) follows from the well known fact that, for $C > 1$, random graphs are almost certainly connected. ∎

Proof (*of Lemma 4.1*). Since

$$dk_0/dp = (y_0 - p\,dy_0/dp)n^2/2y_0^2 \text{ and } p\,dy_0/dp = (pn/2)\cosh^{-2}(pn/2) ,$$

the monotonicity of k_0 follows easily from the simple inequality $\beta < \sinh \beta \cosh \beta$ for $\beta > 0$.

For simplicity, write $\alpha = -1/\varphi''(x_0)$. By definition, $\sigma^2 = nx_0 \dfrac{\alpha}{\alpha + x_0}$, and from [1, (2.6)] we find $\sigma^2 = nx_0(1 - x_0(1 - y_0^2))$. For the first inequality of (4.5), note from [1, (1.3)] that $1 - x(1 - y^2) \ge 2y^2/3$, and that $xy^2 \ge 3(x - 1)/4$. For the second, note from [1, (1.3)] that $3(x - 1) \ge y^2 \ge 1 - x(1 - y^2)$, and multiplication by x gives the result for $4/3 \ge x$; but for $4/3 \le x$, $4(x - 1) \ge x$ trivially.

Equations (4.6) and (4.7) follow from

$$\tanh(z) = z - \frac{1}{3} z^3 + O(z^5)$$

and (4.3). From [1, (2.6)], $\alpha = 2y_0^2/3 + O(y_0^4)$. On the other hand,

$$\frac{k_0}{n} = x_0 - 1 = \frac{y_0^2}{3} + O(y_0^4)$$

and $q_0 \ge n - 1$. Combining these with the definition of σ gives (4.8). ∎

Proof (*of Lemma 4.2*). Let $|r| \le n^{1/32}\sigma$. The assumed lower bound on y_0 implies, for some $c > 0$, $cn^{1/4} < k_0 = O(n \log n)$, and so by (4.5) $r < n^{1/32}k_0^{1/2} = o(k_0)$. Recall that $w_k = \exp(O(k^{-1}))$. We start with (4.1), (1.3), and the following easy estimates:

$$\frac{\binom{N}{q_0+r}}{\binom{N}{q_0}} = \left(\frac{n^2}{2q_0}\right)^r \exp\{-r^2/2q_0 + O(n^{-3/8})\},$$

$$\left(\frac{p}{1-p}\right)^r = p^r \exp\{O(n^{-3/8})\},$$

$$w_{k_0+r} = \exp\{O(n^{-1/4})\},$$

$$a(x_0 + r/n) - a(x_0) = O(n^{-7/16}),$$

$$\frac{(q_0+r)^{1/16}}{n^{9/50}} = O(n^{-1/9})$$

and

$$\frac{r^3}{n^2}\varphi'''(x_0 + \theta r/n) = O(n^{-1/32}\log^{3/2} n) \text{ for } |\theta| \le 1.$$

We have used [1, (3.3), (3.6), (3.11), (3.13)] to see that $a'(x) = O(1/y)$ and that $\varphi'''(x) = O(1/y^4)$. Since

$$\varphi(x_0 + r/n) = \varphi(x_0) + \frac{r}{n}\varphi'(x_0) + \frac{r^2}{2n^2}\varphi''(x_0) + \frac{r^3}{6n^3}\varphi'''(x_0 + \theta r/n)$$

for some $\theta \in (0, 1)$, and since $\exp(-\varphi'(x_0)) = y_0$, we obtain

$$\frac{R_p(n, q_0 + r)}{\hat{R}_p(n, q_0)} = \exp\left\{-\frac{r^2}{2\sigma^2} + O(n^{-1/40})\right\}.$$

The value for $\sum_q R_p(n, q)$ is obtained by restricting r to $|r| \le n^{1/32}\sigma$. The sum for larger r is negligible because

$$\binom{N}{q} p^q (1-p)^{N-q} \exp\{n\varphi(x)\}$$

is log concave with respect to q. (Log concavity follows from the well-known log concavity of $\binom{N}{q} p^q(1-p)^{N-q}$ and the fact that $\varphi'' < 0$.) ∎

Proof (*of Lemma 4.3*). By (4.7) we have $k_0 = (p^2n^3/12)(1 + O(n^{-3/4}))$, and for such small k, [1, (1.9)] is a more convenient presentation of $c(n, n+k)$ than (1.3). From [1, (1.9)]

$$\hat{R}_p(n, q_0)$$
$$= \frac{1}{2} p^{q_0}(1-p)^{N-q_0}\left(1 - \frac{4}{k_0}\right)(3/\pi)^{1/2} n^{n + \frac{1}{2}(3k_0 - 1)}(e/12k_0)^{k_0/2}(1 + O(n^{-3/8})),$$

and we find

$$\frac{R_p(n, q_0+r)}{\hat{R}_p(n, q_0)} = \left(\frac{p}{1-p}\right)^r \frac{w_{k_0+r}}{1-\frac{4}{k_0}} \left(\frac{e}{12}\right)^{r/2} \frac{k_0^{k_0/2}}{(k_0+r)^{(k_0+r)/2}} n^{3r/2}$$
$$\times (1 + O(k_0^{3/2}/n^{1/2})) .$$

We now use the easy estimates (refer to Theorem 1.1 regarding w_k)

$$\frac{k_0^{k_0/2}}{(k_0+r)^{(k_0+r)/2}} = k_0^{-r/2} \exp\{-r/2 - r^2/2\sigma^2 + O(r^3/k_0^2)\}$$

$$(1-p)^r = \exp\{O(rn^{-11/8})\}$$

$$\frac{w_{k_0+r}}{1-\frac{4}{k_0}} = \exp\{O(r/k_0^2)\}$$

and

$$(p^2n^3/12k_0)^{r/2} = 1 + O(rn^{-3/4})$$

to find

$$\frac{R_p(n, q_0+r)}{\hat{R}_p(n, q_0)} = \exp\{-r^2/2\sigma^2 + O(r^3/k_0^2) + O(k_0^{3/2}/n^{1/2})\} .$$

Summing on r, with simple bounds for the tails, completes the proof. ∎

5. BRIDGES

We start by showing the probability that a random edge is not a bridge is approximately y, as given in (1.2). For small k this approximation breaks down, and an asymptotic expansion is found. For large x, $1-y$ is near zero and fails to give the asymptotic probability that a random edge is a bridge; instead, the latter probability is found by showing that almost all bridges, for large x are pendant edges.

In this section we work in the probability space of all connected (n, q) graphs with a distinguished edge, each such graph having a probability of $1/qc(n, q)$. We consider the events

$\mathscr{B}$ the distinguished edge is a bridge;
$\mathscr{B}\mathscr{T}_t$ $\mathscr{B}$ holds and one end of the distinguished edge is a tree with t vertices;
$\mathscr{B}\mathscr{T}$ the union of $\mathscr{B}\mathscr{T}_t$ over all t.

The symbol $\neg$ is used to denote logical negation.

Theorem 5.1. *Let $\epsilon > 0$ be arbitrary. The following results hold uniformly in the ranges indicated.*

(*a*) *For* $1 \le k \le N - n$,

$$\operatorname{Prob}\{\neg\mathscr{B}\} = y(1 + O(1/k) + O(k^{1/16}/n^{9/50})).$$

(*b*) *For* $1 \le k = O(n^{1-\epsilon})$, C_t *as in (3.1) and* $m > 1/\epsilon$, *we have*

$$\operatorname{Prob}\{\neg\mathscr{B}\} \sim \frac{y w_{k-1} e^{-1/2}}{w_k} \left(\frac{k}{k-1}\right)^{(k-1)/2} \exp\left\{-n \sum_{t=2}^{m-1} t C_t k^t / n^t\right\}.$$

(*c*) *For* $k \ge 0$, $\operatorname{Prob}\{\mathscr{B}\mathscr{T}\} = \operatorname{Prob}\{\mathscr{B}\mathscr{T}_1\}(1 + O(xe^{-2x}))$.

(*d*) *For* $k \ge 0$, $\operatorname{Prob}\{\mathscr{B}\mathscr{T}\} = \operatorname{Prob}\{\mathscr{B}\mathscr{T}_1\} e^{x(1-y)}(1 + O((k+1)^{1/16}/n^{9/50}))$.

(*e*) *For* $k = O(n^{2-\epsilon})$,

$$\operatorname{Prob}\{\mathscr{B}\mathscr{T}_1\} = e^x \sqrt{1 - y^2}(1 - q/N)^{n-2} \exp\{(1 - q/N)^{-1}\} \times (1 + O((k+1)^{1/16}/n^{9/50})).$$

$$\operatorname{Prob}\{\mathscr{B}\} = \operatorname{Prob}\{\mathscr{B}\mathscr{T}\} + O(\min(x^2 e^{-x}/n, (nk)^{-1/2})).$$

Remark. We actually derive a big-oh error estimate for (b). See (5.2).

Proof (*of* (*a*)). We have

$$\begin{aligned} \operatorname{Prob}\{\neg\mathscr{B}\} &= \frac{(N - q + 1)c(n, q - 1)}{qc(n, q)} \\ &= \frac{w_{k-1}}{w_k} \exp\{n(\varphi(x - 1/n) - \varphi(x)) + a(x - 1/n) - a)x)\} \\ &\quad \times (1 + O(((k+1)^{1/16}/n^{9/50}))^{-1}, \end{aligned} \tag{5.1}$$

by (1.3). By Taylor's theorem,

$$n(\varphi(x - 1/n) - \varphi(x)) = -\varphi'(x) + \frac{\varphi''(\xi_1)}{2n}$$

and

$$a(x - 1/n) - a(x) = -\frac{a'(\xi_2)}{n}$$

for $x - 1/n < \xi_i < x$. For $k \ge 2$, we use (3.5), (3.10), (3.3), and (3.13) of [1] to bound φ'' and a'. The ratio w_{k-1}/w_k is $1 + O(k^{-1})$, as given in Theorem 1.1. Combining this with (1.6), we obtain (a) for $k \ge 2$. For $k = 1$, use [1, (1.9)]. ∎

Proof (*of* (*b*)). This follows fairly easily from (3.1) and the fact, proved there, that y^2 is a power series in k/n with lead term $3k/n$. By using (3.1), we obtain an error in the exponent which is

$$O(k^{m+1}/n^m) + O(k^{1/2}/n^{1/2}) + O(k^{1/16}/n^{9/50}) . \quad \blacksquare \tag{5.2}$$

Proof (*of* (*c*)). For here and for the proof of (d) we note that, if $\text{Prob}\{\mathcal{BT}_1\} = 0$, then $\text{Prob}\{\mathcal{BT}\} = 0$, so we may limit our attention to $\text{Prob}\{\mathcal{BT}_1\} \neq 0$. Since

$$\text{Prob}\{\mathcal{BT}_1\} = \frac{n(n-1)c(n-1, q-1)}{qc(n, q)} , \tag{5.3}$$

it follows from Theorem 1.1 that this probability is bounded away from zero when x is bounded. Thus we need only consider the case $x \to \infty$. It is easily shown, as was [1, (1.11)], that

$$\text{Prob}\{\mathcal{BT}\} = \text{Prob}\{\mathcal{BT}_1\}\left(1 + \sum_{t=2}^{n-2} \frac{tc(t, t-1)}{t!} \frac{(n-1)_t c(n-t, q-t)}{(n-1)c(n-1, q-1)}\right). \tag{5.4}$$

Let A_t be the general term in the sum. We have

$$A_t = \frac{t^{t-1}}{t!} \frac{\Psi_{-1,t}}{\Psi_{-1,1}} \exp(\Phi_{-1,t} - \Phi_{-1,1}) \frac{1 + b(n-t, q-t)}{1 + b(n-1, q-1)} , \tag{5.5}$$

where b is given by [1, (1.9)], and Φ and Ψ are (see [1, (4.1)])

$$\Phi_{s,t} = (n-t)\varphi\left(\frac{q-t-s-1}{n-t}\right) + a\left(\frac{q-t-s-1}{n-t}\right) - n\varphi(x) - a(x)$$

$$\Psi_{s,t} = \frac{(n-1)_t}{\binom{N}{q}} \binom{\binom{n-t}{2}}{q-t-s-1}.$$

Note $(t+1)^t/(t+1)! = O(e^t)$ by Stirling's formula,

$$\Psi_{-1,t+1}/\Psi_{-1,1} = O((\Psi_{-1,2}/\Psi_{-1,1})^t)$$

by [1, (5.14)] and

$$\Phi_{-1,t+1} - \Phi_{-1,1} \leq O(1) - t\varphi(x) + t(x-1)\varphi'(x)$$

by the concavity of φ. Combining these observations with (5.5), we obtain

$$A_{t+1} = O(1)\left(\frac{\Psi_{-1,2}}{\Psi_{-1,1}} \exp\{1 - \varphi(x) + (x-1)\varphi'(x)\}\right)^t . \tag{5.7}$$

Combining

$$\frac{\Psi_{-1,2}}{\Psi_{-1,1}} = (n-2)\binom{\binom{n-2}{2}}{q-2}\Big/\binom{\binom{n-1}{2}}{q-1}$$

$$\leq \frac{(n-2)(q-1)}{\binom{n-1}{2}}\left(\frac{\binom{n-2}{2}}{\binom{n-1}{2}-1}\right)^{q-2}$$

$$= (1+O(1/n))2xe^{-2x},$$

(1.4), (1.6), and (5.7), we obtain

$$A_{t+1} = O(1)\{(1+O(1/n))xe^{1-x}\sqrt{1-y^2}\}^t. \tag{5.8}$$

The expression inside $\{\cdots\}$ is bounded away from 1 and, by [1, (3.7)], is $O(xe^{-2x})$. This proves that the series in (5.4) can be bounded by a geometric series and (c) follows. ∎

Proof (*of* (*d*)). We will estimate the sum in (5.4). From (5.8), the sum is dominated by a geometric series with ratio bounded away from 1. It follows that we need only sum over $t < C \log n$ for some large enough constant C. From [1, (3.5), (3.13)],

$$\Phi_{-1,t} = -t\varphi(x) + \frac{kt\varphi'(x)}{n} + o(1/n) \tag{5.9}$$

for such t.

From the remarks following (5.8), it follows that the sum is $O(k^{1/16}/n^{9/50})$ for $x > C \log n$ and C sufficiently large. Thus we assume that $x = O(\log n)$. From [1, (5.2)],

$$\Psi_{-1,t} = (2x)^t \exp\{-2xt + o(1/n^{1-\epsilon})\}$$

for t, $x < O(\log n)$. Combining these with (5.5) and (5.9), we obtain

$$\operatorname{Prob}\{\mathscr{B}\mathscr{T}\} = \operatorname{Prob}\{\mathscr{B}\mathscr{T}_1\}(1 + O((k+1)^{1/16}/n^{9/50}))$$
$$\times\left(\sum_{t\geq 1} \frac{t^{t-1}}{t!}(2xe^{-2x-\varphi(x)+(x-1)\varphi'(x)})^{t-1}\right).$$

Using [1, (1.12)] and (1.2), the summation can be simplified to

$$\frac{T(\beta)}{\beta} = \frac{x(1-y)}{\beta} = e^{x(1-y)},$$

which proves (d). ∎

Proof (*of* (*e*)). From (1.6), (5.3), and (5.9),

$$\begin{aligned}\operatorname{Prob}\{\mathscr{B}\mathscr{T}_1\} &= \frac{n(n-1)}{q}\binom{\binom{n-1}{2}}{q-1}\binom{N}{q}^{-1} \\ &\quad\times \exp\{-\varphi(x)+(x-1)\varphi'(x)\}(1+O((k+1)^{1/16}/n^{9/50})) \\ &= e^x\sqrt{1-y^2}\binom{N+2-n}{q}\binom{N}{q}^{-1} \\ &\quad\times(1+O((k+1)^{1/16}/n^{9/50}))\,.\end{aligned}$$

The ratio of binomial coefficients equals

$$\begin{aligned}\prod_{i=0}^{n-3}\frac{N-q-2-i}{N-i} &= \left(1-\frac{q}{N}\right)^{n-2}\prod_{i=0}^{n-3}\frac{1-\dfrac{i+2}{N-q}}{1-\dfrac{i}{N}} \\ &= \left(1-\frac{q}{N}\right)^{n-2}\exp\left\{\sum_{i=0}^{n-3}\frac{i}{N}-\frac{i+2}{N-q}+O(n^3/(N-q)^2)\right\} \\ &= \left(1-\frac{q}{N}\right)^{n-2}\exp\left\{1-\frac{N}{N-q}+O(1/n)\right\} \\ &= \left(1-\frac{q}{N}\right)^{n-2}\exp\{(1-q/N)^{-1}+O(1/n)\}\,.\end{aligned}$$

■

Proof (*of* (*f*)). Writing Δ for $\operatorname{Prob}\{\mathscr{B}-\mathscr{B}\mathscr{T}\}$, we have by [1, (1.11)]

$$\Delta \le \sum_{s=0}^{\frac{k-1}{2}}\sum_{t=1}^{n-1}\binom{n}{t}t(n-t)\,\frac{c(t,t+s)c(n-t,q-t-s-1)}{qc(n,q)}\,. \tag{5.10}$$

Let $T(X)$ be the exponential generating function for rooted trees; that is, $T(X)=\sum_{t=1}^{\infty} tc(t,t-1)X^t/t!$. We use the following, which holds for $0\le X<e^{-1}$, $s\ge 0$:

$$\sum_{t=1}^{n-1}\frac{tc(t,t+s)}{t!}X^t = O(1)(3/2)^s s!\,\frac{T(X)}{(1-T(X))^{3s+2}}\,. \tag{5.11}$$

For $s\ge 1$ this is a consequence of an inequality found in [3], as explained in [1, (10.5)]. For $s=0$ see the explicit formula [1, (2.4)]. Fix δ, $\epsilon_0>0$ and $C_1<1$ so that (see [1, (6.2), (6.4)])

$$\frac{3xy}{2e}(n/k)^{1/2}\le(1-\delta)C_1^{3/2}\,,\qquad x\le 1+\epsilon_0 \tag{5.12}$$

and

$$\frac{x+1}{2}\sqrt{1-y^2} \le 1 - \frac{C_1 k}{n}, \qquad x \le 1 + \epsilon_0 . \tag{5.13}$$

The proof consists of three cases.

Case (i), $0 < k \le \epsilon_0 n$. Using (1.3), (5.6), and $\frac{1}{q}\binom{n}{t} t(n-t) = \frac{1}{x}\frac{t}{t!}(n-1)_t$, we may rewrite (5.10) thus

$$\Delta = O(1) \sum_{s=0}^{\frac{k-1}{2}} \sum_{t=1}^{n-1} \frac{tc(t, t+s)}{t!} \Phi_{s,t} e^{\Psi_{s,t}} . \tag{5.14}$$

Using $stx = O(n^2)$ in [1, (5.21)] to bound $\Phi_{s,t} e^{\Psi_{s,t}}$, and then (5.13),

$$\Delta = O(1) \sum_{s=0}^{\frac{k-1}{2}} (2xy/n)^{s+1} \sum_{t=1}^{n-1} \frac{tc(t, t+s)}{t!} \left(1 - \frac{C_1 k}{n}\right)^t e^{-t} . \tag{5.15}$$

Now apply (5.11) with $X = \left(1 - \frac{C_1 k}{n}\right)\Big/ e$, noting by [1, (8.4)] that $1 - T\left(\left(1 - \frac{C_1 k}{n}\right)\Big/ e\right) \ge (C_1 k/n)^{1/2}$, to obtain

$$\Delta = O(1) \frac{xy}{n} \frac{n}{C_1 k} \sum_{s=0}^{\frac{k-1}{2}} (2xy/n)^s (3/2)^s s! \left(\frac{n}{C_1 k}\right)^{3s/2} .$$

Condition (5.12) implies that $\frac{2xy}{n} \frac{3}{2} \frac{s}{e} \left(\frac{n}{C_1 k}\right)^{3/2} \le 1 - \delta$ for $s \le \frac{k-1}{2}$, and so the sum on s is bounded by $O(1) \sum_{s=0}^{\infty} \sqrt{s}(1-\delta)^s = O(1)$. Hence, $\Delta = O(y/k) = O((kn)^{-1/2})$, the last because $y^2 \le 3(x-1) = 3k/n$. This complete case (i).

Case (ii), $\epsilon_0 n \le k \le 6n \log n$. Let $1 - \delta_0$ be the value of $(x+1)\sqrt{1-y^2}/2$ when $x = \epsilon_0$. Since [1, (8.9)] $(x+1)\sqrt{1-y^2}/2$ decreases when x and y increase, $\delta_0 > 0$. Let $s_0 = C_5 n/x$, where C_5 is fixed and satisfies

$$0 < C_5 \le \frac{e}{3}(1-\delta)\delta_0^{3/2} . \tag{5.16}$$

Separate the bound (5.10) for Δ into two parts

$$\Delta \le \Delta_1 + \Delta_2 ,$$

where Δ_1 is that part of the summation in (5.10) where $0 \le s < s_0$, and Δ_2 is that part where $s_0 \le s \le \frac{k-1}{2}$. We now consider each of Δ_1 and Δ_2 in turn. Again, $stx = O(n^2)$, and so, as in (5.14) and (5.15), we have

$$\Delta_1 = O(1) \sum_{s=0}^{s_0} (2xy/n)^{s+1} \sum_{t=1}^{n-1} \frac{tc(t, t+s)}{t!} \left(\frac{x+1}{2e}\sqrt{1-y^2}\right)^t .$$

Since [1, (8.9)] $(x+1)\sqrt{1-y^2}/2$ decreases as x and y increase, since $T(X) =$

$O(X)$, and since [1, (8.4)]$T((1-\delta_0)/e) \le 1-\sqrt{\delta_0}$, we have by (5.11) with $X=(x+1)\sqrt{1-y^2/2e}$

$$\Delta_1 = O(1)\,\frac{xy}{n}\,x\sqrt{1-y^2}\sum_{s=0}^{s_0}(2xy/n)^s(3/2)^s s!\,\delta_0^{-3s/2}\,.$$

Condition (5.16) implies that $\frac{2xy}{n}\,\frac{3}{2}\,\frac{s}{e}\,\delta_0^{-3/2} \le 1-\delta$ for $s \le s_0$, and so the sum on s is again $O(1)$. Since [1, (3.7)]$1-y^2 = O(e^{-2x})$, we have $\Delta_1 = O(x^2e^{-x}/n)$.

We turn now to Δ_2. Because $\varphi'(x)>0$ and $\varphi''(x)<0$ we have

$$t\varphi\left(\frac{t+s}{t}\right)+(n-t)\varphi\left(\frac{n-t+k-1-s}{n-t}\right) \le n\varphi\left(\frac{q-1}{n}\right) \le n\varphi\left(\frac{q}{n}\right).$$

Since $0<a(x)=O(1)$, we may substitute from (1.3) into (5.10) and find

$$\Delta_2 = O(n^2/q)\binom{N}{q}^{-1}\sum_{s=s_0}^{\frac{k-1}{2}}\sum_{t=1}^{n-1}\binom{n}{t}\binom{\binom{t}{s}}{t+s}\binom{\binom{n-t}{2}}{q-1-t-s}$$

Hence, by [1, (10.1)], $\Delta_2 = O(p(n)2^{-s_0})$, where $p(n)$ is a polynomial in n. This completes case (ii).

Case (iii), $6n\log n \le k \le N-n$. First, note the bound, for $1 \le t \le n-1$,

$$\frac{\binom{N-t(n-t)}{q-1}}{\binom{N}{q-1}} \le \left(\frac{N-t(n-t)}{N}\right)^{q-1} \le \exp\left(-\frac{(q-1)t(n-t)}{N}\right). \tag{5.17}$$

As noted in [1, page 155]

$$\sum_{s=0}^{\frac{k-1}{2}} c(t,t+s)c(n-t,q-1-t-s) \le \binom{N-t(n-t)}{q-1},$$

and so, since $c(n,q) \sim \binom{N}{q}$ by [2], we have from (5.10) and (5.17)

$$\Delta = O(1)\sum_{t=1}^{n-1}\binom{n}{t}t(n-t)\,\frac{\binom{N-t(n-t)}{q-1}}{q\binom{N}{q}}$$

$$= O\ n^2\,\frac{\binom{N}{q-1}}{q\binom{N}{q}}\ \sum_{t=1}^{n/2} n^t\exp\left(-\frac{(q-1)t(n-t)}{N}\right)\Big/t!\,.$$

Because $n\exp(-(q-1)(n-t)/N) \le n\exp(-(q-1)/n) = O(n^{-5})$ for

$1 \le t \le n/2$, the above sum on t is dominated by the first term; also, $q\binom{N}{q} = (N-q+1)\binom{N}{q-1} \ge \binom{N}{q-1}$, and so

$$\Delta = O(n^3 e^{-2x}),$$

since $(n-1)/N = 2/n$. This completes case (iii), and the proof of Theorem 5.1. ∎

6. VERTEX DEGREES AND CUT VERTICES

Recall the convention of Section 1 that a "random vertex" is one selected uniformly from a graph G, with G selected uniformly from the class $C(n, q)$.

We begin the study of vertices by computing the probability that a vertex is a noncut vertex of degree d. We then obtain information about the degrees of cut vertices by showing that removal of a typical cut vertex produces only one component which is not a tree.

Theorem 6.1. *Let $n^{2/9} \le k \le 6n \log n$, let $P(d)$ be the probability that a random vertex is a noncut vertex of degree d and let $\eta = 1/k + k^{1/16}/n^{9/50}$. For $1 \le d = O(\log n)$, we have*

$$P(d) = \frac{\sqrt{1-y^2}}{2ye^x} \frac{(2xy)^d}{d!} (1 + O(\eta)). \tag{6.1}$$

For arbitrary $d > d_0 > 3x$,

$$P(d) < O(1)P(d_0)(2/3 + o(1))^{d-d_0}. \tag{6.2}$$

For n and k as given, the probability that a random vertex is not a cut vertex is

$$e^{-x(1-y)}(1 + O(\eta)). \tag{6.3}$$

Theorem 6.2. *Let $D(d)$ be the probability that a random vertex has degree d and let $D_T(d)$ be the probability that, in addition, removal of the vertex leaves only one component that is not a tree. Let $\eta = 1/k + k^{1/16}/n^{9/50}$. For $n^{2/9} \le k \le 6n \log n$ and $1 \le d = O(\log n)$,*

$$D(d) = D_T(d) + O(\eta) = \frac{\sqrt{1-y^2}}{2ye^x} \frac{(x(1+y))^d - (x(1-y))^d}{d!} + O(\eta). \tag{6.4}$$

Throughout this section, we let

$$H = \begin{cases} n/k^{7/8}, & \text{for } n^{2/9} \le k \le \epsilon_0 n, \\ n^{1/8}, & \text{for } \epsilon_0 n < k \le 6n \log n, \end{cases}$$

ϵ_0 being the constant defined in [1, (6.1)–(6.4)]. We require a technical lemma.

Lemma 6.1. *For $n^{2/9} \le k \le 6n \log n$, $t \le H$ and $s = O(\log n)$, we have*

$$\frac{c(n-t, q-t-s)}{c(n, q)} = \frac{(xe^{-x}\sqrt{1-y^2})}{(n-1)_t} \left(\frac{2xy}{n}\right)^s (1 + O(\eta)), \tag{6.5}$$

where $\eta = \eta(n, k) = 1/k + k^{1/16}/n^{9/50}$

Proof. Since

$$\frac{q-t-s}{n-t} = x + \delta \text{ with } \delta = \frac{(t(x-1) - s}{n-t},$$

we have

$$\frac{c(n-t, q-t-s)}{c(n, q)} = \frac{\binom{\binom{n-t}{2}}{q-t-s}}{\binom{N}{q}} \exp\{a(x+\delta) - a(x)\}$$

$$\times \exp\{\{(n-t)\varphi(x+\delta) - n\varphi(x)\}(1 + O(\eta)). \tag{6.6}$$

We will treat each of the first three factors on the right of (6.6) separately.

In the notation of [1, Lemma 5.2], the first factor of (6.6) is $\Psi_{s-1,t}/(n-1)_t$ and

$$\Psi_{s-1,t} = (2xe^{-2x})^t \frac{(q-t)_s}{(N-B-q+s)_s}$$

$$\times \exp\{O(tx^2/n) + O(t^2(x-1)/n) + O(t^2x^3/n^2)\}, \tag{6.7}$$

where $B = t(n - t/2 - 3/2)$. The exponential in (6.7) is $\exp(O(\eta))$. We have

$$q^s > (q-t)_s > q^s\left(1 - \frac{t+s}{q}\right)^s = q^s \exp\left\{O\left(\frac{st+s^2}{q}\right)\right\} = q^s \exp(O(\eta))$$

and so $(q-t)_s = q^s(1 + O(\eta))$. Since

$$n^{2s}/2^s > (N-B-q+s)_s > (N-B-q)^s$$

and

$$\frac{2(N-B-q)}{n^2} = 1 - \frac{(2t+1) + 2x - t(t+3)/n}{n},$$

it follows that

$$(N-B-q+s)_s = n^{2s}2^{-s}(1 + O(\eta)).$$

Using these estimates with (6.7), we have

$$\frac{\binom{\binom{n-t}{2}}{q-t-s}}{\binom{N}{q}} = \frac{(2xe^{-2x})^t}{(n-1)_t}\left(\frac{2x}{n}\right)^s(1+O(\eta)). \tag{6.8}$$

Since $a'(x)$ is $O(n^{1/2}/k^{1/2})$ for $y \leq 3/4$ and bounded elsewhere by [1, (3.2), (3.8), (3.13)], we have

$$a(x+\delta) - a(x) = O(\eta). \tag{6.9}$$

We have

$$(n-t)\varphi(x+\delta) - n\varphi(x) = -t\varphi(x) + (n-t)\delta\varphi'(x) + O(n\delta^2\varphi''(\xi)), \tag{6.10}$$

where ξ is between x and $x+\delta$. By [1, (3.5), (3.10)], $O(n\delta^2\varphi''(\xi)) = O(\eta)$. Thus by (1.4), (1.6), (6.10), and the definition of δ,

$$\exp\{(n-t)\varphi(x+\delta) - n\varphi(x)\} = \left(\frac{e^x\sqrt{1-y^2}}{2}\right)^t y^s(1+O(\eta)). \tag{6.11}$$

The lemma follows from (6.8), (6.9), and (6.11). ∎

Proof (*of Theorem 6.1*). To make a vertex v a noncut vertex, we construct a connected graph on the remaining vertices and connect v to it with d edges. Thus

$$P(d) = \binom{n-1}{d}\frac{c(n-1, q-d)}{c(n,q)}. \tag{6.12}$$

As in the proof of Lemma 6.1, $(n-1)_d = n^d(1+O(\eta))$. Using (6.5) with $t=1$ and $s = d-1$, we obtain from (6.12)

$$P(d) = \frac{n^d}{d!}\frac{x\sqrt{1-y^2}}{ne^x}\left(\frac{2xy}{n}\right)^{d-1}(1+O(\eta)).$$

This proves (6.1).

We now prove (6.2). Substituting (1.3) into (6.12) and using the boundedness of a and the positivity of φ', it follows that

$$\begin{aligned}\frac{P(d)}{P(d_0)} &= O(1)\frac{\binom{n-1}{d}}{\binom{n-1}{d_0}}\frac{\binom{N-n+1}{q-d}}{\binom{N}{q-d_0}}\\ &\leq O(1)\prod_{i=1}^{d-d_0}\frac{(n-d_0-i)}{(d_0+i)}\frac{(q+i-d)}{(N+d-q-i)}.\end{aligned}$$

Clearly each fraction inside the product is at most $nq/d_0(N-q)$, which, since $d_0 > 3x$, is at most $2/3 + o(1)$. This proves (6.2).

Using (6.1) and (6.2), sum $P(d)$ over $d > 0$ to obtain

$$\sum P(d) = \frac{\sqrt{1-y^2}}{2ye^x}(e^{2xy} - 1)(1 + O(\eta)).$$

Now use (1.2) to obtain (6.3). ∎

Proof (*of Theorem 6.2*). The plan of the proof is as follows. For the stated range of d, show that $D_T(d)$ equals the expression on the right side of (6.4). Define $\Delta(d)$ to be $D_T(d)$ in that range and 0 otherwise. Next, conclude that $\Sigma\, \Delta(d) = 1 + O(\eta)$. Finally, since $D(d) - \Delta(d) \geq 0$ and $\sum_{d=1}^{\infty} D(d) = 1$, conclude that $D(d) - \Delta(d) \leq O(\eta)$.

Let $f_j(t)$ be the number of t vertex labeled forests that contain exactly j rooted labeled trees. The exponential generating function for f_j is $(T(z))^j/j!$, where $T = ze^T$ is the exponential generating function for rooted labeled trees.

By an argument like that for (6.12), it is easy to see that

$$D_T(d) = \sum_{\substack{0 \leq j < d \\ 0 \leq t < n}} \binom{n-1}{t} f_j(t) \binom{n-t-1}{d-j} \frac{c(n-t-1,\, q-t-(d-j)))}{c(n,q)}. \tag{6.13}$$

For $t \leq H$, we can replace $\binom{n-t-1}{d-j}$ with $\frac{n^{d-j}}{(d-j)!}$. Also, we can apply Lemma 6.1 with t replaced by $t+1$ and s by $d-j-1$. Thus, a term in (6.13) is

$$\begin{aligned} &\frac{n^{d-j}(n-1)_t}{t!(d-j)!} f_j(t) \frac{(xe^{-x}\sqrt{1-y^2})^{t+1}}{(n-1)_{t+1}} \left(\frac{2xy}{n}\right)^{d-j-1} (1 + O(\eta)) \\ &\quad = \frac{\sqrt{1-y^2}}{2ye^x} \frac{(2xy)^{d-j}}{(d-j)!} \frac{f_j(t)}{t!} \left(\frac{x\sqrt{1-y^2}}{e^x}\right)^t (1 + O(\eta)). \end{aligned}$$

We wish to sum this for $0 \leq t < n$. We may bound the tails of the sum, where $t > H$, by using [1, Lemma 5.4] and arguments like those in [1, Section 8]. This gives

$$D_T(d) = \sum_{0 \leq j < d} \frac{\sqrt{1-y^2}}{2ye^x} \frac{(2xy)^{d-j}}{(d-j)!} \frac{T(\beta)^j}{j!} (1 + O(\eta)), \tag{6.14}$$

where $\beta = xe^{-x}\sqrt{1-y^2}$ is the same as in [1, (1.13)]. From [1, (1.12)], $T(\beta) = x(1-y)$. Using this in (6.14), we obtain

$$D_T(d) = \frac{\sqrt{1-y^2}}{2ye^x d!} \sum_{j=0}^{d-1} \binom{d}{j} (2xy)^{d-j} (x(1-y))^j (1 + O(\eta)),$$

which easily gives (6.4) for $D_T(d)$. Note that the formula vanishes for $d = 0$.

Summing this on $d \geq 0$, with easy estimates for $d = \Omega(\log n)$, gives

$$\sum \Delta(d) = \frac{\sqrt{1-y^2}}{2ye^x}(\exp\{x(1+y)\} - \exp\{x(1-y)\})(1+O(\eta)).$$

Using (1.2) and a bit of algebra, this reduces to

$$\sum \Delta(d) = (1 + O(\eta)). \qquad \blacksquare$$

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