The Asymptotic Number of Labeled Connected Graphs with a Given Number of Vertices and Edges

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ABSTRACT

Let $c(n, q)$ be the number of connected labeled graphs with n vertices and $q \le N = \binom{n}{2}$ edges. Let $x = q/n$ and $k = q - n$. We determine functions $w_k \sim 1$, $a(x)$ and $\varphi(x)$ such that $c(n, q) \sim w_k \binom{N}{q} e^{n\varphi(x) + a(x)}$ uniformly for all n and $q \ge n$. If $\epsilon > 0$ is fixed, $n \to \infty$ and $4q > (1 + \epsilon)n \log n$, this formula simplifies to $c(n, q) \sim \binom{N}{q} \exp(-ne^{-2q/n})$. On the other hand, if $k = o(n^{1/2})$, this formula simplifies to $c(n, n + k) \sim \frac{1}{2} w_k (3/\pi)^{1/2} (e/12k)^{k/2} n^{n+(3k-1)/2}$.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper all graphs are labeled, and an (n, q) graph is one having n vertices and q edges. Let $c(n, q)$ equal the number of connected (n, q) graphs. Our object

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is to study the asymptotic behavior of $c(n, q)$ as $n, q \to \infty$. Throughout we let

$$N = \binom{n}{2}, \quad k = q - n \text{ and } x = q/n = 1 + k/n\,. \tag{1.1}$$

The function $y = y(x) > 0$ is defined implicitly by

$$2xy = \log\left(\frac{1+y}{1-y}\right), \tag{1.2}$$

for $1 < x < \infty$. We can extend it continuously to $x = 1$ by defining $y(1) = 0$. Another way to express the relationship between x and y is

$$x = 1 + \frac{y^2}{3} + \frac{y^4}{5} + \cdots\,. \tag{1.3}$$

This given, we can state our main result.

Theorem 1. *With k, N, x, and y defined as above, the number of connected (n, q) graphs, $c(n, q)$, is given by*

$$c(n, q) = \binom{N}{q} e^{a(x)} \left(\frac{2e^{-x} y^{1-x}}{\sqrt{1-y^2}}\right)^n \left[1 + O\left(\frac{1}{k}\right) + O\left(\frac{k^{1/16}}{n^{9/50}}\right)\right] \tag{1.4}$$

as $n \to \infty$, uniformly for $0 < k \le N - n$, where

$$a(x) = x(x+1)(1-y) + \log(1 - x + xy) - \tfrac{1}{2}\log(1 - x + xy^2)\,. \tag{1.5}$$

The binomial coefficient $\binom{N}{q}$ is the total number of (n, q) graphs, so the rest of the expression on the right side of (1.4) may be interpreted as the probability of connectedness.

Erdős and Rényi [5] proved a formula equivalent to (1.4) when $x - \frac{1}{2}\log n$ is bounded below. Wright [14] proved a formula like (1.4) but without the $O(1/k)$ term when $k = o(n^{1/3})$. We will use Wright's result in Corollary 1 to eliminate our $O(1/k)$ term. In Corollaries 2 and 3, we will show that the forms given by Erdős and Rényi and by Wright are valid over a larger range. Stepanov [10] estimated the probability of connectedness for random graphs with edge probability p. Although the expected number of edges given that the graph is connected is significantly larger than pN [2], Stepanov's results can be used to obtain a close upper bound on $c(n, q)$ [9]. The formula for connected graphs is closely related to that for weakly connected digraphs [3].

We wish to thank Tomasz Łuczak for suggesting the simple proof of Lemma 3.7, Lambert Meertens for suggesting the simple proof of Lemma 3.4, and Maple for integrating $a'(x)$.

Corollary 1. *Define* $w_0 = \pi/\sqrt{6}$ *and*

$$w_k = \frac{\pi\Gamma(k)d_k(8/3)^{1/2}}{\Gamma(3k/2)}\left(\frac{27k}{8e}\right)^{k/2} \text{ for } k>0\,, \tag{1.6}$$

where d_k *is given by the recursion*

$$d_1 = \frac{5}{36}\,,\quad d_{k+1} = d_k + \sum_{h=1}^{k-1} \frac{d_h d_{k-h}}{(k+1)\binom{k}{h}} \text{ for } k>0\,. \tag{1.7}$$

For $k \geq 0$, *we have*

$$c(n,q) = w_k \binom{N}{q}\left(\frac{2e^{-x}y^{1-x}}{\sqrt{1-y^2}}\right)^n e^{a(x)}\left[1 + O\left(\frac{(k+1)^{1/16}}{n^{9/50}}\right)\right], \tag{1.8}$$

where the indeterminate value of y^{1-x} *at* $x = 1$ *is taken to be* 1 *and that of* $a(1)$ *is taken to be* $2 + \frac{1}{2}\log(3/2)$.

Corollary 2. *If* w_k *is given as in Corollary* 1, *then*

$$c(n, n+k) = \tfrac{1}{2}\, w_k(3/\pi)^{1/2}(e/12k)^{k/2}n^{n+(3k-1)/2}$$
$$[1 + O(k^2/n) + O((k+1)^{1/16}/n^{9/50})]\,, \tag{1.9}$$

uniformly for $0 \leq k = O(n^{1/2})$.

Corollary 3. *If* $\epsilon > 0$ *is fixed, then*

$$c(n,q) \sim \binom{N}{q}\exp(-ne^{-2q/n}) \tag{1.10}$$

uniformly for $4q > (1+\epsilon)n\log n$.

The definition of y^{1-x} at $x = 0$ in Corollary 1 is not arbitrary. By (1.3), it is the value that makes y^{1-x} continuous at 0. Wright [14] introduced d_k and (1.7) and proved that $d = \lim_{k\to\infty} d_k$ exists. Meertens [7] and Voblyi [11] proved that $d = 1/2\pi$. At the cost of some complication in the expressions, (1.10) and (1.9) can be extended to wider ranges of x [2].

Our proof of Theorem 1 is based on the following recursive formula for $c(n,q)$:

$$qc(n,q) = (N - q + 1)c(n, q-1)$$
$$+ \frac{1}{2}\sum_{t=1}^{n-1}\sum_{s=-1}^{k}\binom{n}{t}t(n-t)c(t, t+s)c(n-t, q-t-s-1)\,. \tag{1.11}$$

This identity follows from counting in two ways the connected (n, q) graphs having a distinguished edge. The left side of (1.11) corresponds to starting with a connected (n, q) graph and then choosing one of its edges to distinguish. The two quantities on the right side of (1.11), the first a single term, the second a summation, correspond to adding a distinguished edge to an existing graph in two ways: First we may add the distinguished edge to a connected $(n, q-1)$ graph, choosing a pair of vertices not already connected; second we may choose one vertex from a connected $(n-t, q-t-s-1)$ graph and a second vertex from a disjoint connected $(t, t+s)$ graph, between which we add the distinguished edge. These two cases on the right side of (1.11) correspond respectively to the cases where the distinguished edge is not, and is, a bridge.

We now state Theorem 2, which describes the asymptotic behavior of $c(n, q)$ in different terms than Theorem 1 does. In fact, (1.12) and (1.17) below are the equations which arise "naturally" when one assumes formula (1.19) for $c(n, q)$, and then uses recursion (1.11) to find conditions on $\varphi(x)$ and $a(x)$. We will prove Theorem 2 and derive the other results from it.

Theorem 2. *Let $\varphi(x)$ be the solution of the differential equation*

$$1 = e^{-\varphi'(x)} + \frac{1}{x} T(\beta) \tag{1.12}$$

where $\varphi'(1) = +\infty$ and $\beta = \beta(x)$ is defined by

$$\beta(x) = 2xe^{-2x+(x-1)\varphi'(x)-\varphi(x)} \tag{1.13}$$

and $T(u)$ is the exponential generating function for labeled and rooted trees:

$$T(u) = \sum_{t=1}^{\infty} \frac{tc(t, t-1)}{t!} u^t . \tag{1.14}$$

Let $U(u)$ be the exponential generating function for labeled and rooted unicyclic graphs:

$$U(u) = \sum_{t=3}^{\infty} \frac{tc(t, t)}{t!} u^t . \tag{1.15}$$

Let

$$g_1(x) = -2(x-1)(x+1) \text{ and } g_2(x) = -\frac{(x-1)(2x-1)}{2x} \tag{1.16}$$

and let $a(x)$ be the solution of the differential equation

$$0 = \left[\frac{1}{2}\varphi''(x) - a'(x)\right] e^{-\varphi'(x)} + \frac{1}{x}\beta^2 T''(\beta)\left[g_2(x) + \frac{1}{2}\varphi''(x)(x-1)^2\right] + \frac{1}{x}\beta T'(\beta)\left[g_1(x) + a'(x)(x-1) + \frac{1}{2}\varphi''(x)(x-1)^2\right] + 2e^{-\varphi'(x)}U(\beta) \tag{1.17}$$

which satisfies the boundary condition

$$a(1) = \tfrac{1}{2}\log(3/2) + 2 . \tag{1.18}$$

Then, if $b(n, k)$ is defined by the identity

$$c(n, n+k) = \binom{N}{n+k} \exp[n\varphi(x) + a(x)][1 + b(n, k)] , \qquad (1.19)$$

we will have

$$b(n, k) = O(1/k) \text{ uniformly for } 0 < k \leq n^{1/5} , \qquad (1.20a)$$

and

$$b(n, k) = O(k^{1/16}/n^{9/50}) \text{ uniformly for } n^{1/5} < k \leq N - n . \qquad (1.20b)$$

It is not immediately evident that the boundary condition $\varphi'(1) = +\infty$ determines $\varphi(x)$. This will follow in the process of proving Lemma 2.1.

The functions $\varphi(x)$ and $a(x)$ are rather complicated, so we have included some plots at the end of the paper. We have also included some plots of error estimates given by

$$[1 + \epsilon(n, q)]c(n, q) = \binom{N}{q} \exp[a(x) + n\varphi(x)]$$

and

$$[1 + \epsilon_2(n, q)]c(n, q) = w_k \binom{N}{q} \exp[a(x) + n\varphi(x)] .$$

These suggest that our approximations are rather good.

The rest of the paper is organized as follows. In the next section, we prove the equivalence of Theorems 1 and 2. Section 3 begins with a variety of easy estimates involving $y(x)$, $\varphi(x)$ and $a(x)$. Then we (a) show that the theorems plus Wright's result imply the corollaries, (b) use Wright's result to prove Theorem 2 for $k \leq n^{2/9}$, and (c) obtain an upper bound for $c(n, q)$. In Section 4 we state Lemma 4.1 and show that it is all that is needed to complete the proof of Theorem 2. Section 5 contains three lemmas that estimate two functions appearing in the statement of Lemma 4.1. Sections 6–10 are devoted to proving Lemma 4.1. To do this, we divide the range $n^{2/9} \leq k \leq N - n$ into subranges. The points of division are $k = \epsilon_0 n$ and $k = 6n \log n$, where ϵ_0 is a constant that will be specified later.

Throughout the remainder of this paper, the symbol T with no explicitly stated argument will be understood to mean $T(\beta(x))$. Likewise, U will always mean $U(\beta(x))$ and expressions like T' will mean $dT(u)/du$ with u then set to $\beta(x)$. We will often use standard asymptotic methods without explicit reference. Information on these methods can be found in de Bruijn [4] and Bender [1].

2. THEOREMS 1 AND 2 ARE EQUIVALENT

We require some preliminary results.

Lemma 2.1. *We have*

$$\frac{dx}{dy} = \frac{1 - x + xy^2}{y(1 - y^2)} = \frac{2y}{3} + \frac{4y^3}{5} + \cdots, \tag{2.1}$$

$$u = T(u)\, e^{-T(u)}, \tag{2.2}$$

$$uT'(u) = \frac{T(u)}{1 - T(u)}, \tag{2.3}$$

$$U(u) = \frac{\frac{1}{2} T(u)^3}{[1 - T(u)]^2}. \tag{2.4}$$

Between $y(x)$ *introduced in* (1.2) *and* $\varphi(x)$ *defined by* (1.12) *we have the relations*

$$y = e^{-\varphi'(x)}, \tag{2.5}$$

$$\varphi''(x) = \frac{-1}{y\, dx/dy} = -\frac{1 - y^2}{1 - x + xy^2} \tag{2.6}$$

$$= \frac{-1}{y(2y/3 + 4y^3/5 + \cdots)}, \tag{2.7}$$

$$e^{\varphi(x)} = \frac{2e^{-x} y^{1-x}}{\sqrt{1 - y^2}}. \tag{2.8}$$

Proof. Equation (2.1) follows easily from (1.2) and (1.3).

Equation (2.2) follows from the well-known identity

$$T(u) = ue^{T(u)}. \tag{2.9}$$

(See, e.g., [8, p. 77].) Differentiating (2.9) with respect to u and rearranging, we obtain (2.3). Equation (2.4) was proved by Wright [13, 15].

Let $f(x) = e^{-\varphi'(x)}$. We will show that $f(x)$ satisfies (2.1) with y replaced by f. Rearrange (1.12) as

$$x - xf = T \tag{2.10}$$

and differentiate both sides of the equation with respect to x. Rewrite the resulting right side by using (2.3) evaluated at β and the logarithmic derivative of (1.13). The latter will contain neither $\varphi(x)$ nor $\varphi'(x)$, but will contain $\varphi''(x)$, which can be replaced by $-(df/dx)/f$, from (2.5). In this manner, both sides of (2.10), after differentiation with respect to x, are rational functions of df/dx, x, f, and T. Multiplying through by $xf(1 - T)$ to clear functions and eliminating T with (2.10), we obtain

$$(1-x+xf)\Big(xf-x^2f\,\frac{df}{dx}-xf^2\Big)=x(1-f)\Big(f-2xf-x(x-1)\,\frac{df}{dx}\Big).$$

Solving for dx/df, we find that

$$\frac{dx}{df}=-\frac{x}{f}-\frac{1}{f(f^2-1)},$$

which is (2.1) with y replaced by f. Since $f(1)=e^{-\infty}=0$ by the definition of φ, we have $f=y$. Hence, (2.5) is verified. Equations (2.6) and (2.7) follow easily.

Next we consider (2.8). Let $u=\beta(x)$ in (2.2) and then use (1.13) to eliminate $\beta(x)$ on the left and (2.10) to eliminate T on the right. Use (2.5) to eliminate $\varphi'(x)$. The resulting equation can be solved for $e^{\varphi(x)}$ and (2.8) results. ∎

Lemma 2.2. *Theorem* 1 *is equivalent to Theorem* 2.

Proof. By (2.8), it suffices to show that $a(x)$ given by (1.5) is the same as $a(x)$ given by (1.17) and (1.18). To distinguish between the two versions of $a(x)$ in our proof, we will use $\alpha(x)$ to denote the value given by (1.5).

It follows easily from (1.3) that, for any $\epsilon>0$

$$y=\sqrt{3}\Big(\frac{k}{n}\Big)^{1/2}\Big(1+O\Big(\frac{k}{n}\Big)\Big) \tag{2.11}$$

uniformly for $0\le k/n<\epsilon$. Using this fact, it is easy to see that $\lim_{x\to 1}\alpha(x)$ is $\frac{1}{2}\log(3/2)+2$.

A messy algebraic calculation, which we present in summary, shows that $\alpha(x)$ satisfies (1.17). Rearrange (1.17) so that all occurences of $a'(x)$ are on one side, use (2.3) to express $\beta T'$ and β^2T'' as rational functions in T, use (2.4), and replace $e^{-\varphi'(x)}$ by y to obtain

$$\begin{aligned} a'(x)\Big(y-\frac{(x-1)T}{x(1-T)}\Big)&=\varphi''(x)\Big(\frac{y}{2}+\frac{(x-1)^2T}{2x(1-T)^3}\Big)\\ &\quad+\frac{T}{(1-T)^3}\Big(\frac{(2x-1)(x-1)}{-2x^2}\Big)\\ &\quad-\frac{T}{1-T}\Big(\frac{(x-1)(4x^2+2x+1)}{2x^2}\Big)+\frac{yT^3}{(1-T)^2}. \end{aligned} \tag{2.12}$$

The quantity multiplying $a'(x)$ on the left is $x(1-x+xy^2)/x(1-T)$. Divide both sides of (2.12) by this expression, multiply both sides by dx/dy, given by (2.1), and use (2.6) to eliminate $\varphi''(x)$. This leads to

$$\begin{aligned} a'(x)\,\frac{dx}{dy}&=\frac{1-T}{-2(1-x+xy^2)}+\frac{(x-1)^2(1-y)}{-2y(1-T)^2(1-x+xy^2)}\\ &\quad+\frac{(2x-1)(x-1)}{-2xy(1-T)^2(1+y)}+\frac{(1-x)(4x^2+2x+1)}{2xy(1+y)}\\ &\quad+\frac{x^3(1-y)^2}{(1-T)(1+y)}. \end{aligned} \tag{2.13}$$

Use (2.10) to eliminate T. From (1.5) we can compute $d\alpha/dy$ and verify that this equals the previous expression. ∎

We remark for future use that, after some rearrangement, (2.13) can be written as

$$a''(x)\frac{dx}{dy} = -x^2 + \frac{4x^2 - 2x - 1}{2y} - \frac{2x^2}{1 - x + xy}\frac{1-y}{1+y} + \frac{xy}{1 - x + xy^2} \tag{2.14}$$

3. SOME EASY ESTIMATES

Lemma 3.1. *We have, uniformly for* $0 < y \le 3/4$, *or equivalently for* $1 < x \le \frac{2}{3}\log 7$, *the following*:

$$\sqrt{1-y^2} = 1 - 3k/2n + O(k^2/n^2)\,, \tag{3.1}$$

$$a(x) = O(1)\,, \tag{3.2}$$

$$a'(x) = O(n^{1/2}/k^{1/2})\,, \tag{3.3}$$

$$a''(x) = O(n^{3/2}/k^{3/2})\,, \tag{3.4}$$

$$\varphi''(x) = O(1/y^2) = O(n/k)\,, \tag{3.5}$$

$$\varphi'''(x) = O(1/y^4) = O(n^2/k^2)\,. \tag{3.6}$$

Proof. Equation (3.1) is an easy consequence of (1.3). Some calculations with (1.5) and (1.3) show that $a(x)$ has a power series expansion about $y = 0$ and so $d^k a(x)/(dy)^k$ is bounded for all k. Equation (3.2) follows immediately and (3.3) and (3.4) follow upon use of the chain rule, (2.1) and (2.11). Equations (3.5) and (3.6) are easily obtained in a similar manner from (2.7). ∎

Lemma 3.2. *As* $x \to \infty$,

$$1 - y = 2\exp[-2x + O(xe^{-2x})] \sim 2e^{-2x}\,, \tag{3.7}$$

$$x^m(1-y) = O(e^{-\lambda x}) \text{ for every } \lambda < 2\,, \tag{3.8}$$

where the constant implied by the big-oh depends on λ *and* m,

$$\frac{dy}{dx} = O(1-y)\,, \tag{3.9}$$

$$\varphi''(x) = O(1-y)\,, \tag{3.10}$$

$$\varphi'''(x) = O(1-y)\,, \tag{3.11}$$

$$a(x) = O(x^2(1-y))\,, \tag{3.12}$$

$$a'(x) = O(x^2(1-y))\,, \tag{3.13}$$

$$a''(x) = O(x^2(1-y))\,. \tag{3.14}$$

Proof. From (1.2) with $\delta = 1 - y$,

$$e^{-2x} = \frac{(1-y)^{1/y}}{(1+y)^{1/y}} = \frac{1-y}{2}\left(\frac{(\delta/2)^{-\delta}}{1-\delta/2}\right)^{1/y} = \frac{1}{2}\,\delta\,\exp(O(\delta\log\delta))\,. \tag{3.15}$$

Since $\delta\log\delta \to 0$, it follows that $e^{-2x} \sim \delta/2$. Combining this with (3.15), we obtain

$$\tfrac{1}{2}(1-y) = \exp[-2x + O(\delta\log\delta)] = \exp[-2x + O(xe^{-2x})]\,.$$

This proves (3.7) and (3.8) and so

$$1 - x + xy = 1 - x(1-y) \sim 1\,. \tag{3.16}$$

Combining (2.1) and (3.16) gives (3.9).

We will say that $f(x, y)$ is $\mathrm{Rat}_{r,m}$ for $r \geq 0$, $m \geq 0$ if $f = O(x^r(1-y)^m)$ and there are polynomials $P(x, y)$ and $Q(x, y)$ such that $f = P/Q$ and $\lim_{x\to\infty} Q(x, y)$ exists and is not zero (of course, $y \to 1$ as $x \to \infty$). We now prove

$$\text{if } f \text{ is } \mathrm{Rat}_{r,m}, \text{ then } \frac{df}{dx} \text{ is } \mathrm{Rat}_{r,m}\,. \tag{3.17}$$

Let $\delta = 1 - y$ and write P and Q as polynomials in x and δ, say $\Sigma\, p_{i,j}x^i\delta^j$ and $\Sigma\, q_{i,j}x^i\delta^j$. By the assumption on $\lim Q$ and (3.8) we see that $q_{0,0} \neq 0$ and $q_{i,0} = 0$ for $i > 0$. By this, (3.7) and the assumption on f, $p_{i,j} = 0$ if $j < m$ or $j = m$ and $i < r$. It follows easily using this and (3.8) that for some t

$$\begin{aligned} P_xQ - PQ_x &= O(x^{r-1}(1-y)^m)O(1) + O(x^r(1-y)^m)O(x^t(1-y)) \\ &= O(x^r(1-y)^m)\,. \end{aligned} \tag{3.18}$$

Also, for some u,

$$\begin{aligned} P_yQ - PQ_y &= O(x^r(1-y)^{m-1})O(1) + O(x^r(1-y)^m)O(x^u) \\ &= O(x^r(1-y)^{m-1})\,. \end{aligned}$$

Combining this with (3.18) and (3.9) gives (3.17).

From (2.6), (2.1) and (3.9), $\varphi''(x)$ is a rational function of x and y and (3.10) holds. Now (3.11) follows from (3.17). Equation (3.12) follows from (1.5) and (3.8). From (2.14) and (3.8),

$$a'(x)\,\frac{dx}{dy} = O(x^2) \text{ and is a rational function of } x \text{ and } y.$$

By (1.5) and (2.1), $a'(x)$ is $\mathrm{Rat}_{2,1}$, implying (3.13). We have (3.14) by (3.17). ∎

In the next lemma, we estimate $n\varphi(x) + a(x)$ for large x.

Lemma 3.3. *Fix $\epsilon > 0$. When $4x > (1+\epsilon)\log n$, Theorem 2 implies Corollary 3. When $q > (\frac{1}{2}+\epsilon)n\log n$,*

$$\exp[n\varphi(x) + a(x)] \sim 1 . \tag{3.19}$$

Proof. From (1.2) and (2.8),

$$\begin{aligned} e^{\varphi(x)} &= \frac{2}{1+y}\exp[-x(1-y)-(x-1)\log y] \\ &= \exp\{\tfrac{1}{2}(1-y) + O((1-y)^2) - x(1-y) - (x-1)[-(1-y)+O((1-y)^2)]\} \\ &= \exp[-\tfrac{1}{2}(1-y) + O(x(1-y)^2)] . \end{aligned}$$

Combining this with (3.12) and using (3.7), we have

$$\begin{aligned} \exp[n\varphi(x)+a(x)] &= \exp[-ne^{-2x}(1+O(xe^{-2x}) + O(nxe^{-4x}) + O(x^2e^{-2x})] \\ &= \exp[-ne^{-2x} + O(nxe^{-4x}) + O(x^2e^{-2x})] . \end{aligned}$$

Since $4x > (1+\epsilon)\log n$ and since x^2e^{-x} is a decreasing function for such x, it follows that the $O(\ )$ terms are $o(1)$. This implies the lemma. ∎

Lemma 3.4. *The numbers d_k given by (1.7), satisfy $d_k = 1/2\pi + O(1/k)$.*

Proof. The following simple proof was suggested by Meertens [7]. It can be sharpened to give the asymptotic expansion suggested by Wright [14, Sec. 6], which begins

$$d_k = \frac{1}{2\pi}\left[1 - \frac{5}{18k} - \frac{155}{648k^2} + O\left(\frac{1}{k^3}\right)\right], \tag{3.20}$$

and it can be used for any $d_1 \in (0, \frac{1}{4})$, with different constants appearing in (3.20).

Define the formal power series

$$D(x) = \sum_{k=1}^{\infty} (k-1)!d_kx^k$$

$$\exp(D(x)) = J(x) = 1 + \sum_{k=1}^{\infty} (k-1)!j_kx^k .$$

From (1.7), $x^2D'' + (2x-1)D' + x^2(D')^2 - d_1 = 0$. Thus

$$x^2J'' + (2x-1)J' + d_1J = 0 .$$

Equating coefficients, we find that

$$j_i = d_1 \text{ and } j_{k+1} = j_k\left(1 + \frac{d_1}{k(k+1)}\right) \text{ for } k \geq 1 .$$

Hence

$$j_\infty = d_1 \prod_{i=1}^{\infty}\left(1+\frac{d_1}{i(i+1)}\right) = \frac{1}{\pi}\cos\left(\frac{1}{2}\pi\sqrt{1-4d_1}\right) = \frac{1}{2\pi},$$

$$j_k = j_\infty\left[\prod_{i=1}^{k}\left(1+\frac{d_1}{i(i+1)}\right)\right]^{-1} = \frac{1}{2\pi} + O\left(\frac{1}{k}\right),$$

and $0 < j_k < j_\infty$. It follows from [12, Theorem 2] that for any fixed $R > 0$

$$d_n = \sum_{k=0}^{R-1} \frac{c_k j_{n-k}}{(n-1)_k} + O(n^{-R}),$$

where $\Sigma\, c_k x^k = 1/J(x)$ and $(\ )_k$ denotes the falling factorial. ∎

We now prove (1.20) for sufficiently small k. This will be a beginning of a proof by induction of the full statement of (1.20b).

Lemma 3.5. *Theorem 2 is true for* $1 \le k \le n^{2/9}$ *and* (1.9) *follows from* (1.8).

Proof. We begin by estimating terms in (1.19). Then we use Wright's and Meertens' results to estimate $b(n, k)$. The claim concerning Corollary 2 then follows easily.

If we use (2.8) and let $x = 1 + k/n$, we have

$$e^{n\varphi(x)} = \left(\frac{2e^{-x}y^{1-x}}{\sqrt{1-y^2}}\right)^n = 2^n e^{-n-k} y^{-k}(1-y^2)^{-n/2}. \tag{3.21}$$

Using (3.1), we obtain

$$(1-y^2)^{n/2} = \left[1 - \frac{3k}{2n} + O\left(\frac{k^2}{n^2}\right)\right]^n = e^{-3k/2}\left[1 + O\left(\frac{k^2}{n}\right)\right]$$

and so (3.21) becomes

$$e^{n\varphi(x)} = 2^n e^{-n+k/2} y^{-k}[1 + O(k^2/n)].$$

Referring to (2.11), we may rewrite this as

$$e^{n\varphi(x)} = 2^n 3^{-k/2} e^{-n+k/2} k^{-k/2} n^{k/2}[1 + O(k^2/n)]. \tag{3.22}$$

We easily have

$$(n+k)!\binom{N}{n+k} = \left(\frac{n^2}{2}\right)^{n+k}\left(1-\frac{1}{n}\right)^{n+k}\prod_{j=0}^{n+k-1}\left(1-\frac{j}{N}\right),$$

$$\left(1-\frac{1}{n}\right)^{n+k} = e^{-1}\left[1 + O\left(\frac{k}{n}\right)\right],$$

and

$$\prod_{j=0}^{n+k-1}\left(1-\frac{j}{N}\right)=e^{-1}\left[1+O\left(\frac{k}{n}\right)\right].$$

Combining these results gives

$$(n+k)!\binom{N}{n+k}=n^{2n+2k}2^{-n-k}e^{-2}[1+O(k/n)]. \tag{3.23}$$

By Stirling's formula,

$$(n+k)!=n^{n+k}e^{-n}\sqrt{2\pi n}[1+O(k^2/n)]. \tag{3.24}$$

From (1.18) and (3.3),

$$a(1+k/n)=\tfrac{1}{2}\log(3/2)+2+O(k^{1/2}/n^{1/2}). \tag{3.25}$$

Multiplying (3.22), (3.23), and the exponential of (3.25), and dividing by (3.24), we have, assuming $k^2=o(n)$,

$$\binom{N}{n+k}\exp[n\varphi(x)+a(x)]=\frac{1}{2}\sqrt{3/\pi}n^{n+(3k-1)/2}\left(\frac{e}{12k}\right)^{k/2}[1+O(k^{1/2}/n^{1/2})]$$
$$\times[1+O(k^2/n)]. \tag{3.26}$$

Wright [14, Theorem 2] shows that $c(n,n)\sim\sqrt{\pi/8}n^{n-1/2}$ and

$$c(n,n+k)=n^{n+(3k-1)/2}\frac{\pi^{1/2}3^k\Gamma(k)}{2^{(5k-1)/2}\Gamma(3k/2)}d_k[1+O(k^{3/2}/n^{1/2})] \tag{3.27}$$

uniformly for $0<k=o(n^{1/3})$, where d_k is given by (1.7). If Stirling's formula is used for $\Gamma(k)$ and $\Gamma(3k/2)$ in (3.27) and the result is compared with (3.26), we find that (1.8) implies (1.9). If we also use Lemma 3.4 and the definition of $b(n,k)$ in (1.19), we find that, for $k\le n^{2/9}$,

$$b(n,k)=O(1/k)+O(k^{3/2}/n^{1/2})+O(k^2/n). \tag{3.28}$$

For $k\le n^{1/5}$, $O(1/k)$ dominates in (3.28). For $n^{1/5}\le k\le n^{2/9}$, $O(k^{3/2}/n^{1/2})$ dominates in (3.28) and is dominated by $O(k^{1/16}/n^{9/50})$. ∎

Lemma 3.6. *Theorem 2 with Wright's result implies Corollaries 1 and 2.*

Proof. By Lemma 3.5, it suffices to prove (1.8). For $k\ge n^{2/9}$, it follows from the theorem. For $k<n^{2/9}$, it follows from (1.6), (3.27), and (3.26). ∎

In order to carry out the induction, we need some sort of crude upper bound for $c(n,q)$. This result is needed to bound parts of the sum in (1.11). The exact form of the bound is somewhat arbitrary.

Lemma 3.7. *For $n>0$ and $n<q\le N$,*

$$c(n,q)\le O(n^{3/2})\binom{N}{q}e^{n\varphi(x)}. \tag{3.29}$$

We originally produced a lengthy proof of lemma with $O(n^{3/2})$ replaced by q^2. Tomasz Łuczak told us about this simple probabilistic method [6] for proving this type of result. Later, Boris Pittel [9] showed us how to use Stepanov's upper bound on the probability of connectedness to obtain a factor of $O((n\log n)^{1/2})$.

Proof. We use $f=\Theta(g)$ to indicate that $|f/g|$ is bounded away from 0 and ∞. We may assume that $x<\frac{1}{2}\log n+O(1)$, since otherwise $e^{n\varphi(x)}=\Theta(1)$. Thus, from (3.7) and $y'(x)>0$, we may assume that $(1-y)n\ge 1$.

Following Łuczak, we use the edge probability model $G(m,p)$ of random graphs and compute the expected number of components having n vertices and q edges. Since this number cannot exceed m/n, we have

$$\binom{m}{n}c(n,q)p^q(1-p)^{n(m-n)+N-q}\le\frac{m}{n}. \tag{3.30}$$

Set $p=2xy/n$ and $m=(1+y)n/2y+\delta$, where $0\le\delta<1$ and m is an integer. We use $z!$ for $\Gamma(z+1)$. Note that $(z+\delta)!=\Theta(z!z^\delta)$ for $z>1/2$. Using this and (1.2), we have

$$\begin{aligned}\binom{\frac{(1+y)n}{2y}+\delta}{n}&=\frac{((1+y)n/2y+\delta)!}{n!((1-y)n/2y+\delta)!}\\&=\Theta(1)\left(\frac{(1+y)n/2y}{(1-y)n/2y}\right)^\delta\binom{(1+y)n/2y}{n}\\&=\Theta(1)\left(\frac{1+y}{1-y}\right)^\delta\binom{(1+y)n/2y}{n}\\&=\Theta(e^{2\delta xy})\binom{(1+y)n/2y}{n}.\end{aligned}$$

From this and the easily derived $(1-p)^{n\delta}\sim\exp(-2\delta xy)$, it follows that we can neglect δ with a multiplicative error of $\Theta(1)$. Rearranging (3.30), using Stirling's formula to eliminate the binomial coefficient, and using Taylor series to expand $\log(1-p)$, we obtain

$$\begin{aligned}c(n,q)=\frac{O(1)}{y}&\left[\sqrt{\frac{n(1-y)}{1+y}}\left(\frac{[(1-y)/2y]^{(1-y)/2y}}{[(1+y)/2y]^{(1+y)/2y}}\right)^n\right]\\&\times\left(\frac{n}{2xy}\right)^{xn}\exp\left[\left(\frac{2xy}{n}+\frac{2x^2y^2}{n^2}\right)\left(\frac{n^2(1-y)}{2y}+N-nx\right)\right].\end{aligned}\tag{3.31}$$

Note that

$$\binom{N}{q} = \binom{N}{nx} = \Theta(1)\,\frac{(e/nx)^{nx}}{(nx)^{1/2}}\,N^{nx}\prod_{i<nx}\left(1-\frac{i}{N}\right)$$
$$= \frac{\Theta(1)}{(nx)^{1/2}}\left(\frac{e(n-1)}{2x}\right)^{nx}\exp(-x^2) \tag{3.32}$$

and, by (1.2),

$$\frac{[(1-y)/2y]^{(1-y)/2y}}{[(1+y)/2y]^{(1+y)/2y}} = \frac{2y[(1-y)/(1+y)]^{1/2y}}{(1-y^2)^{1/2}} = \frac{2ye^{-x}}{\sqrt{1-y^2}}\,. \tag{3.33}$$

Combining (3.31)–(3.33), rearranging, and using (2.8) and (3.8), we obtain

$$\frac{c(n,q)}{\binom{N}{q}} = O(1)\,\frac{n\sqrt{x(1-y)}}{y}\left(\frac{1}{ey}\right)^{nx}\left(\frac{2y}{\sqrt{1-y^2}}\right)^{n}\exp[x(1+x)(1-y)]$$
$$= O(1)(n/y)\,e^{n\varphi(x)}\,.$$

The lemma follows from (2.11). ∎

4. THE REDUCTION OF THEOREM 2 TO A CRUCIAL LEMMA

Our plan is to prove (1.20b) by a double induction on k and n. This hinges on Lemma 4.1 which we will soon state. In this section we show that Lemma 4.1 implies (1.20b). The proof of the lemma itself occupies the remainder of this paper.

It will be useful to have the following notation for two frequently occuring complicated expressions.

$$\Phi_{s,t} = (n-t)\varphi\left(\frac{q-t-s-1}{n-t}\right) + a\left(\frac{q-t-s-1}{n-t}\right) - n\varphi(x) - a(x) \tag{4.1a}$$

and

$$\Psi_{s,t} = \frac{(n-1)_t}{\binom{N}{q}}\binom{\binom{n-t}{2}}{q-t-s-1}, \tag{4.1b}$$

with $(\;)_t$ denoting the falling factorial and the dependence on n and k of both Φ and Ψ being understood.

We begin by converting (1.11) into a recursion for $b(n, k)$. To do this, divide both sides of (1.11) by $q\binom{N}{q}\exp[n\varphi(x) + a(x)]$ and replace $c(n, q)$, $c(n, q-1)$, $c(n-t, q-t)$ and $c(n-t, q-t-1)$ by their equivalents as given in (1.19). Making use of the relations

$$\frac{1}{q}\binom{n}{t}t(n-t) = \frac{n}{q}\,\frac{t}{t!}\,(n-1)_t, \text{ and } \frac{n}{q} = \frac{1}{x}\,,$$

we find that

$$
\begin{aligned}
1+b(n,k) = {} & \exp(\Phi_{0,0})[1+b(n,k-1)] \\
& + \frac{1}{x}\sum_{t=1}^{n-1} \frac{tc(t,t-1)}{t!}\,\Psi_{-1,t}\exp(\Phi_{-1,t})[1+b(n-t,k)] \\
& + \frac{1}{x}\sum_{t=1}^{n-1} \frac{tc(t,t)}{t!}\,\Psi_{0,t}\exp(\Phi_{0,t})[1+b(n-t,k-1)] \\
& + \frac{1}{2}\sum_{s=1}^{k-2}\sum_{t=1}^{n-1}\binom{n}{t}t(n-t)\,\frac{c(t,t+s)c(n-t,q-t-s-1)}{q\binom{N}{q}\exp[n\varphi(x)+a(x)]}\,.
\end{aligned}
\tag{4.2}
$$

We rewrite this as

$$
1+b(n,k) = A + S_{-1} + S_0 + S\,, \tag{4.3}
$$

defining A to be the first term on the right of (4.2) and S_{-1}, S_0, and S to be first, second, and third summations on the right of (4.2), respectively. S_i results from combining the $s=i$ and $s=k-i-1$ summations on the right of (1.11) and S results from the remainder of the double summation on the right of (1.11).

Using (4.2) to bound $b(n,k)$ is complicated by the appearance of the quantities of interest on both sides. We use a double induction on k and n, the crux of which is expressed in Lemma 4.1 below. Before stating this lemma, we define the relevant partial order for our double induction and also two functions of x whose origin will become apparent as the proof progresses.

Recall the definitions (1.13) and (1.16) of β, g_1 and g_2.

Definitions.

$$
P(x) = \tfrac{1}{2}\varphi''(x) - a'(x) \tag{4.4}
$$

$$
\begin{aligned}
Q(x) = {} & \frac{\beta T'}{x}\,[g_1(x) + \tfrac{1}{2}(x-1)^2\varphi''(x) + (x-1)a'(x)] \\
& + \frac{\beta^2 T''}{x}\,[g_2(x) + \tfrac{1}{2}\,(x-1)^2\varphi''(x)]\,.
\end{aligned}
\tag{4.5}
$$

We use $<$ *to denote the product order on the product* $\mathbb{N}\times\mathbb{N}$, *that is,*

$$
(n_1,k_1) < (n_2,k_2)
$$

provided that either $n_1 \le n_2$ *and* $k_1 < k_2$, *or* $n_1 < n_2$ *and* $k_1 \le k_2$.

Lemma 4.1. *Let* A, S_{-1}, S_0, *and* S *be the functions of* n *and* k *defined in the remarks following* (4.3) *above, and let* P *and* Q *be the functions of* x *given in* (4.4) *and* (4.5). *Then there is an* n_0 *and a* $C_2 > 0$ *such that, for all* $C \ge 1$, *all* $n \ge n_0$ *and all* k *in the range* $n^{2/9} < k \le N-n$, *we have either*
(a) for some $(\nu,\kappa) < (n,k)$ *with* $\kappa \ge \nu^{1/5}$

$$|b(\nu, \kappa)| > \frac{C\kappa^{1/16}}{\nu^{9/50}}$$

or (b) all four of the following inequalities hold:

$$\left|A - e^{-\varphi'(x)}\left(1 + \frac{1}{n} P(x)\right)\right| \le \frac{1}{C_2} k^{-3/2} n^{-1/2} + \frac{Ck^{1/16}}{n^{9/50}} \left(e^{-\varphi'(x)}\left(1 + \frac{1}{n} P(x) - \frac{1}{16k}\right) + \frac{1}{C_2} k^{-3/2} n^{-1/2}\right), \tag{4.6a}$$

$$\left|S_{-1} - \left(\frac{1}{x} T + \frac{1}{n} Q(x)\right)\right| \le \frac{1}{C_2} k^{-3/2} n^{-1/2} + \frac{1}{C_2} C \exp(-C_2 n^{1/9}) + \frac{Ck^{1/16}}{n^{9/50}} \left(\frac{1}{x} T + \frac{1}{n} Q(x) + \frac{9}{50} \frac{1}{xn} \beta T' + \frac{1}{C_2} k^{-3/2} n^{-1/2}\right), \tag{4.6b}$$

$$\left|S_0 - \frac{2}{n} e^{-\varphi'(x)} U\right| \le \frac{1}{C_2} k^{-3/2} n^{-1/2} + \frac{1}{C_2} C \exp(-C_2 n^{1/9}) + \frac{Ck^{1/16}}{n^{9/50}} \left(\frac{2}{n} e^{-\varphi'(x)} U + \frac{1}{C_2} k^{-3/2} n^{-1/2}\right), \tag{4.6c}$$

$$|S| \le \frac{1}{C_2} Ck^{1/16} k^{-3/2} n^{-1/2}. \tag{4.6d}$$

Now, before proving Lemma 4.1 in the next several sections, we will first see how it implies (1.20b). We begin with a technical inequality.

Lemma 4.2. *There is a universal constant $B > 0$ such that for $k > 0$*

$$\frac{(1/16)\, e^{-\varphi'(x)}}{k} - \frac{(9/50)\beta T'}{xn} > \begin{cases} B/k, & \text{if } k \ge 4n/5, \\ B/(kn)^{1/2}, & \text{if } k \le 4n/5. \end{cases} \tag{4.7}$$

Proof. Suppose that $k \ge 4n/5$; i.e., $x \ge 1.8$. It follows easily from (2.1) that $x(1-y)$ is a decreasing function. Since $y(1.8) > 0.9$, we have $x(1-y) < 0.18$. Hence, by (1.14) and (2.3),

$$\beta T' = \frac{T}{1-T} = \frac{x(1-y)}{1 - x(1-y)} < 0.22.$$

Since $xn = q > k$ and $(1/16)(9/10) > (9/50)(0.22)$, the proof is complete for $k \ge 4n/5$.

Now suppose that $k \le 4n/5$. By (1.3) and $0 < y < 1$,

$$\frac{9}{50} y^2(x-1) - \frac{9}{50}(x-1) + \frac{1}{16} y^2 = \frac{9}{50} \sum_{k=2}^{\infty} \frac{y^{2k}}{2k-1} - \frac{9}{50} \sum_{k=1}^{\infty} \frac{y^{2k}}{2k+1} + \frac{1}{16} y^2 > -\frac{(9/50)y^2}{3} + \frac{1}{16} y^2 = \frac{y^2}{400}. \tag{4.8}$$

Since $y < 1$, (4.8) yields

$$\left(\frac{9}{50}-\frac{1}{16}\right)y(x-1)-\frac{9}{50}(x-1)+\frac{1}{16}xy^2$$
$$>\left(\frac{9}{50}-\frac{1}{16}\right)y^2(x-1)-\frac{9}{50}(x-1)+\frac{1}{16}xy^2>\frac{y^2}{400}.$$

Rearranging,

$$(1-x-xy)\left(\frac{1}{16}y\right)-\frac{9}{50}(1-y)(x-1)>y^2/400. \tag{4.9}$$

Since $x(1-y)$ is a decreasing function, $1-x+xy>0$ when $x>1$. Thus we may divide (4.9) by $x-1$ and $1-x+xy$ to obtain

$$\frac{(1/16)y}{x-1}-\frac{(9/50)(1-y)}{(1-x+xy)}>\frac{y^2}{400(x-1)(1-x+xy)}. \tag{4.10}$$

By (2.5), (2.3), and (2.10), the left sides of (4.7) and (4.10) differ by just a factor of n. Since $1-x+xy=1-x(1-y)<1-(1-y)=y$,

$$\frac{y^2}{(x-1)(1-x+xy)}>\frac{y}{x-1}. \tag{4.11}$$

When $x\le 1.8$, $y\le 0.94$ and, by (1.3), $k/n=x-1<y^2/(1-y)<17y^2$ and so $y=\Omega(\sqrt{k/n})$. [We define $f=\Omega(g)$ to be the same as $g=O(f)$.] Thus $y/(x-1)=\Omega(\sqrt{n/k})$. This, with (4.10) and (4.11), completes the proof of the lemma.

■

Proof of Theorem 2, *Assuming Lemma* 4.1. For the purposes of this proof, define

$$f(n,k)=\begin{cases}(kn)^{1/2}, & \text{if } k\ge 4n/5,\\ k, & \text{if } k\le 4n/5.\end{cases}$$

Let n_0 and C_2 be as given by Lemma 4.1. We must show that there is a C sufficiently large that

$$|b(n,k)|<Ck^{1/16}/n^{9/50} \text{ for } n^{1/5}<k\le N-n. \tag{4.12}$$

Choose $n_1\ge n_0$ sufficiently large that

$$\exp(-C_2n^{1/9})<\frac{k^{1/16}}{n^{9/50}}k^{-3/2}n^{-1/2} \tag{4.13}$$

for $n\ge n_1$ and $1\le k\le N-n$. Choose $n_2\ge n_1$ sufficiently large that

$$\frac{Bf(n,k)}{2}>\frac{5+n^{9/50}}{C_2} \tag{4.14}$$

for $n \geq n_2$ and $k \geq n^{2/9}$. (This B is the constant given in Lemma 4.2.) Finally, choose C sufficiently large that

i. (4.12) holds for all pairs (n, k) with $n^{1/5} < k \leq n^{2/9}$, which can be done by Lemma 3.5.
ii. (4.12) holds for $n < n_2$, a finite set of pairs (n, k).
iii. $3/C_2 \leq (Ck^{1/16}/n^{9/50})Bf(n, k)/2$ for $n \geq n_2$ and $k \geq n^{2/9}$.

Recall the defining relations (1.12) and (1.17) for $\varphi(x)$ and $a(x)$, the latter of which can be stated more succinctly using $P(x)$ and $Q(x)$:

$$1 = e^{-\varphi'(x)} + \frac{1}{x} T , \tag{4.15}$$

$$0 = e^{-\varphi'(x)}P(x) + Q(x) + 2e^{-\varphi'(x)}U . \tag{4.16}$$

Thus

$$\begin{aligned} |b(n, k)| &= \left|(1 + b(n, k)) - \left(e^{-\varphi'(x)} + \frac{1}{x}T\right)\right. \\ &\qquad \left. - \frac{1}{n}\left(e^{-\varphi'(x)}P(x) + Q(x) + 2e^{-\varphi'(x)}U\right)\right| \\ &\leq \left|A - e^{-\varphi'(x)}\left(1 + \frac{1}{n} P(x)\right)\right| + \left|S_{-1} - \left(\frac{1}{x} T + \frac{1}{n} Q(x)\right)\right| \\ &\qquad + \left|S_0 - \frac{2}{n} e^{-\varphi'(x)}U\right| + |S| , \end{aligned}$$

using (4.3) and the triangle inequality.

The proof that (4.12) holds for all pairs (n, k) satisfying $n^{1/5} \leq k \leq N - n$ is completed by double induction once we verify that, when $n \geq n_2$, the sum of the four quantities appearing on the right sides of (4.6a)–(4.6d) is less than $Ck^{1/16}/n^{9/50}$. By (4.13), it suffices to verify this inequality after each of the two occurences of $C \exp(-C_2 n^{-1/9})$ has been replaced by $(Ck^{1/16}/n^{9/50})k^{-3/2}n^{-1/2}$. Making these replacements and transferring all items involving C to the right side, we find that we must verify

$$\begin{aligned} \frac{3}{C_2} k^{-3/2}n^{-1/2} \leq \frac{Ck^{1/16}}{n^{9/50}} &\left[1 - e^{-\varphi'(x)}\left(1 + \frac{1}{n} P(x) - \frac{1}{16k}\right)\right. \\ &\qquad - \frac{1}{x} T - \frac{1}{n} Q(x) - \frac{9}{50} \frac{1}{xn} \beta T' - \frac{2}{n} e^{-\varphi'(x)}U \\ &\qquad \left. - \frac{1}{C_2} (5 + n^{9/50})k^{-3/2}n^{-1/2}\right] . \end{aligned}$$

After multiplication by $k^{3/2}n^{1/2}$ and use of relations (4.15) and (4.16), this becomes

$$\frac{3}{C_2} \le \frac{Ck^{1/16}}{n^{9/50}} \left[\left(\frac{(1/16)\, e^{-\varphi'(x)}}{k} - \frac{(9/50)\beta T'}{xn} \right) k^{3/2} n^{1/2} - \frac{5 + n^{9/50}}{C_2} \right].$$

By Lemma 4.2 it suffices to have

$$\frac{3}{C_2} \le \frac{Ck^{1/16}}{n^{9/50}} \left[Bf(n, k) - \frac{5 + n^{9/50}}{C_2} \right],$$

and then by (4.14) it suffices to have

$$\frac{3}{C_2} \le \frac{Ck^{1/16}}{n^{9/50}} \frac{Bf(n, k)}{2} .$$

This is condition iii on the choice of C and so the proof of Theorem 2 assuming Lemma 4.1 is complete. ∎

5. THREE LEMMAS FOR $\Psi_{s,t}$ AND $\Phi_{s,t}$

Recall the definitions (4.1) of $\Psi_{s,t}$ and $\Phi_{s,t}$. In this section we prove three lemmas about them which will be used in the following sections. We need to have the following result on binomial coefficients ready.

Lemma 5.1. *Let $\lambda \ge 1$ be an integer. Then,*

$$\sum_{\substack{j+h=\lambda \\ j \ge h}} \frac{1}{j} \binom{j}{h} 2^{j-h} (-1)^h = \frac{2}{\lambda} . \tag{5.1}$$

Proof. Call the summation $f(\lambda)$. By using

$$\frac{1}{j} \binom{j}{h} = \frac{1}{h} \binom{j-1}{h-1} \text{ and } \lambda = j + h ,$$

it is easy to prove that $\lambda f(\lambda) = g(\lambda) - g(\lambda - 2)$, where

$$g(\lambda) = \sum_{\substack{j+h=\lambda \\ j \ge h}} \binom{j}{h} 2^{j-h} (-1)^h .$$

Then, by using

$$\binom{j}{h} = \binom{j-1}{h} + \binom{j-1}{h-1},$$

it is easy to prove that $g(\lambda) = 2g(\lambda - 1) - g(\lambda - 2)$. Examining the first few values, by induction we have $g(\lambda) = \lambda + 1$. ∎

Lemma 5.2. *If $\epsilon > 0$ is fixed but arbitrary, then*

$$\Psi_{s,t} = (2x)^t \frac{(q-t)_{s+1}}{(N-B-q+s+1)_{s+1}}$$
$$\times \exp\left[-2xt + \frac{t}{n} g_1(x) + \frac{t(t-1)}{n} g_2(x) + O\left(\frac{(x-1)t^3}{n^2}\right) + O\left(\frac{x^3t^2}{n^2}\right)\right] \quad (5.2)$$

uniformly for

$$0 < k \le n^{2-\epsilon}, \; -1 \le s \le k, \text{ and } 1 \le t \le n^{1-\epsilon} - 1,$$

where $g_1(x)$ and $g_2(x)$ are given by (1.16) and

$$B = B(t) = (n-1)t - (t^2 + t)/2.$$

Remark 1. We have expressed the expression inside the exponential in terms of t/n and $t(t-1)/n$ instead of t/n and t^2/n to facilitate later summation.

Remark 2. The constants implied by the big-oh notation depend on ϵ.

Proof of Lemma 5.2. We first express the desired quantity as products of lengths t, q, and $s+1$. Let $M = \binom{n-t}{2}$. We start with

$$\frac{\binom{M}{q-t-s-1}}{\binom{N}{q}} = \frac{(M)_{q-t-s-1}}{(N)_q} \frac{q!}{(q-t-s-1)!}. \quad (5.3)$$

Next we have

$$\frac{q!}{(q-t-s-1)!} = (q)_{t+s+1} = (q)_t (q-t)_{s+1}$$

and, with B given as in the statement of the lemma, we see that

$$B = N - M - t = O(n^{2-\epsilon})$$

and that $q = O(n^{2-\epsilon}) = o(N-B)$. Hence

$$(M)_{q-t-s-1} = (N-B-t)_{q-t-s-1} = \frac{(N-B-t)_{q-t}}{(N-B-q+s+1)_{s+1}}$$
$$= \frac{(N-B)_q}{(N-B)_t} \frac{1}{(N-B-q+s+1)_{s+1}}. \quad (5.4)$$

Thus, altogether,

$$\Psi_{s,t} = \frac{(n-1)_t (q)_t}{(N-B)_t} \frac{(N-B)_q}{(N)_q} \frac{(q-t)_{s+1}}{(N-B-q+s+1)_{s+1}}. \quad (5.5)$$

Let $m \ge 2$ be a fixed positive integer sufficiently large that

$$t^{m-1} \leq n^{m-2} \text{ and } t^m \leq (n-1)^{m-1}$$

for $t < n^{1-\epsilon}$ and $t < n-1$. We shall use the obvious relation $(t/n)^j = O(t^2/n^2)$, for $j \geq 2$, several times, without explicit mention after the first use.

We now treat the first two fractions appearing in (5.5) individually. First,

$$\frac{(N-B)_q}{(N)_q} = \prod_{i=0}^{q-1}\left(1 - \frac{B}{N-i}\right) = \exp\left[\sum_{i=0}^{q-1} \log\left(1 - \frac{B}{N-i}\right)\right]$$

$$= \exp\left[-\sum_{i=0}^{q-1}\sum_{j=1}^{m} \frac{1}{j}\left(\frac{B}{N-i}\right)^j + O\left(\frac{qB^{m+1}}{N^{m+1}}\right)\right]. \tag{5.6}$$

Interchange the order of summation, replace $B/(N-i)$ by $(B/N)[1 + i/N + O(i^2/N^2)]$ when $j = 1$ in the resulting outer summation, and $[B/(N-i)]^j = (B/N)^j[1 + O(i/N)]$ by $(B/N)^j + O(iB^2/N^3)$ when $j \geq 2$ and use $\Sigma_{i=0}^{q-1} i = q^2/2 + O(q)$ to obtain

$$\frac{(N-B)_q}{(N)_q} = \exp\left[-\frac{qB}{N} - \frac{q^2B}{2N^2} - q\sum_{j=2}^{m}\frac{1}{j}\left(\frac{B}{N}\right)^j + O\left(\frac{q^3B}{N^3}\right) + O\left(\frac{q^2B^2}{N^3}\right) + O\left(q\left(\frac{B}{N}\right)^{m+1}\right)\right].$$

Since $q^2B^2/N^3 = O((xt/n)^2)$, $q^3B/N^3 = O(x^3t/n^2)$, and

$$q(B/N)^{m+1} = O(xt^{m+1}/n^m) = O(xt^2/n^2)$$

by the choice of m, we find

$$\frac{(N-B)_q}{(N)_q} = \exp\left[-q\sum_{j=1}^{m}\frac{1}{j}\left(\frac{B}{N}\right)^j - \frac{q^2B}{2N^2} + O\left(\frac{x^3t^2}{n^2}\right)\right]. \tag{5.7}$$

If we use $\Sigma_{i=0}^{t-1} i = t^2/2 - t/2$, $\Sigma_{i=0}^{t-1} i^j = t^{j+1}/(j+1) + O(t^j)$ for $2 \leq j \leq m$, expand the logarithm to m terms as in (5.6), and make a few easy estimates, we find

$$(n-1)_t = (n-1)^t \exp\left[\frac{t}{2n} - \sum_{j=1}^{m}\frac{1}{j(j+1)}\frac{t^{j+1}}{n^j} + O\left(\frac{t^2}{n^2}\right)\right] \tag{5.8}$$

and

$$(q)_t = q^t \exp\left[\frac{t}{2xn} - \sum_{j=1}^{m}\frac{1}{j(j+1)}\frac{t^{j+1}}{x^jn^j} + O\left(\frac{t^2}{n^2}\right)\right]. \tag{5.9}$$

Next,

$$(N-B)_t = N^t \prod_{i=0}^{t-1}\left(1 - \frac{B+i}{N}\right)$$

$$= N^t \exp\left[-\sum_{j=1}^{m}\sum_{i=0}^{t-1}\frac{1}{j}\left(\frac{B+i}{N}\right)^j + O\left(\frac{tB^{m+1}}{N^{m+1}}\right)\right].$$

Replacing $[(B+i)/N]^j$ by $(B/N)^j[1+O(i/B)]$,

$$(N-B)_t = N^t \exp\left[-t\sum_{j=1}^{m}\frac{1}{j}\left(\frac{B}{N}\right)^j + O\left(\frac{t^2}{n^2}\right)\right]. \tag{5.10}$$

We are ready to multiply together (5.7), (5.8), and (5.9), and divide by (5.10). This will give

$$\frac{(n-1)_t(q)_t(N-B)_q}{(N-B)_t(N)_q} = \frac{(n-1)^t q^t}{N^t}\exp(\gamma), \tag{5.11}$$

where γ is the result of collecting the various arguments of the exponential in the four equations. The quantity raised to the tth power on the right, $(n-1)q/N$, is $2x$. As for γ, we first note that for bounded j

$$\begin{aligned}\frac{B}{N} &= \frac{t(n-1)}{n(n-1)/2} - \frac{(t^2+t)/2}{n(n-1)/2}\\ &= \frac{2t}{n} - \frac{t^2}{n^2} + O\left(\frac{t}{n^2}\right)\\ &= \frac{2t}{n}\left(1-\frac{t}{2n}\right)\left[1+O\left(\frac{1}{n}\right)\right],\end{aligned}$$

$$\left(\frac{B}{N}\right)^j = \left[\sum_{h=0}^{j}\binom{j}{h}\left(\frac{t}{n}\right)^{j+h}2^{j-h}(-1)^h\right]\left[1+O\left(\frac{1}{n}\right)\right], \tag{5.12}$$

$$\frac{qB}{N} = 2xt - \frac{x(t^2+t)}{n} + O\left(\frac{xt^2}{n^2}\right),$$

$$\frac{q^2B}{N^2} = \frac{4x^2t}{n} + O\left(\frac{x^2t^2}{n^2}\right),$$

$$\frac{tB}{N} = \frac{2t^2}{n} - \frac{t^3}{n^2} + O\left(\frac{t^2}{n^2}\right).$$

Taking the four equations (5.7), (5.8), (5.9), and (5.10) in that order and treating the $j=1$ term of each summation separately, we find

$$\begin{aligned}\gamma = &\left(-2xt + \frac{x(t^2+t)}{n} - \frac{2x^2t}{n}\right) + \left(\frac{t}{2n} - \frac{t^2}{2n}\right) + \left(\frac{t}{2xn} - \frac{t^2}{2xn}\right)\\ &+\left(\frac{2t^2}{n} - \frac{t^3}{n^2}\right) - \sum_{j=2}^{m}\sum_{h=0}^{j}\frac{1}{j}\binom{j}{h}2^{j-h}(-1)^h\left(\frac{xt^{j+h}}{n^{j+h-1}} - \frac{t^{j+h+1}}{n^{j+h}}\right)\\ &-\sum_{j=2}^{m}\frac{1}{j(j+1)}\frac{t^{j+1}}{n^j}(1+x^{-j}) + O\left(\frac{x^3t^2}{n^2}\right).\end{aligned} \tag{5.13}$$

Collecting and rearranging [note that the first $j=2$, $h=0$ term of the double summation equals $-2xt^2/n$, and that the $-t^3/n^2$ term of (5.13) has been absorbed into the second summation on λ below when $j=h=1$], we have

$$\gamma = -2xt + \frac{t}{n}\left(-2x^2 + x + \frac{1}{2} + \frac{1}{2x}\right) + \frac{t^2}{n}\left(-x + \frac{3}{2} - \frac{1}{2x}\right)$$
$$-\sum_{\lambda=2}^{m-1} \frac{t^{\lambda+1}}{n^\lambda}\left[x \sum_{\substack{j+h=\lambda+1\\ j\geq h}} \frac{1}{j}\binom{j}{h}2^{j-h}(-1)^h + \frac{1+x^{-\lambda}}{\lambda(\lambda+1)}\right.$$
$$\left. - \sum_{\substack{j+h=\lambda\\ j\geq h}} \frac{1}{j}\binom{j}{h}2^{j-h}(-1)^h\right] + \sum_{\lambda=m}^{2m} \frac{t^{\lambda+1}}{n^\lambda}\,O(x) + O\left(\frac{x^3t^2}{n^2}\right).$$

In the first summation on the right, the coefficient of $t^{\lambda+1}/n^\lambda$ is a rational function of x which grows linearly as $x\to\infty$. According to (5.1), this function vanishes when $x=1$. Thus this function is $O(x-1)$. Hence, since $g_1 - g_2 = (-2x^2 + x + 1/2 + 1/2x)$,

$$\gamma = -2xt + \frac{t}{n}\,g_1(x) + \frac{t(t-1)}{n}\,g_2(x) + O\left(\frac{(x-1)t^3}{n^2}\right) + O\left(\frac{x^3t^2}{n^2}\right).$$

Combining this last equation, (5.11), and (5.5), we obtain the lemma. ∎

Lemma 5.3. *For fixed n, k, and s with $-1\leq s<k$, the numbers $\Psi_{s,t}$ are log concave; that is*

$$(\Psi_{s,t})^2 \geq \Psi_{s,t-1}\Psi_{s,t+1} \text{ for } 1\leq t<n-1. \tag{5.14}$$

Proof. Letting $J=\binom{n-t}{2}$, the desired result is equivalent to

$$\left(1+\frac{1}{n-t-1}\right)\left(1+\frac{1}{q-t-s-1}\right)\left((J)_{q-t-s-1}\right)^2$$
$$\geq (J+(n-t))_{q-t-s}(J-(n-t-1))_{q-t-s-2}\,. \tag{5.15}$$

As a first step toward proving (5.15), we claim that

$$\left(1+\frac{1}{n-t-1}\right)\left(1+\frac{1}{q-t-s-1}\right)J^2 \geq (J+(n-t))$$
$$\times (J+(n-t)-(q-t-s)+1)\,. \tag{5.16}$$

Dividing both side by J^2, one sees that this last inequality is equivalent to

$$\left(1+\frac{1}{n-t-1}\right)\left(1+\frac{1}{q-t-s-1}\right) \geq \left(1+\frac{2}{n-t-1}\right)\left(1-\frac{2(k-s-1)}{(n-t)(n-t-1)}\right). \tag{5.17}$$

When the latter is multiplied out, $-4(k-s-1)/[(n-t)(n-t-1)^2]$ is ignored on the right, and like terms are cancelled from each side, we find that (5.17) is implied by

$$\frac{2(k-s-1)}{(n-t)(n-t-1)} \geq \frac{k-s-1}{(n-t-1)(q-t-s-1)} .$$

The latter is certainly true since $(q-t-s-1)=(n-t)+(k-s-1)$, and $k-s-1 \geq 0$.

Thus (5.16) is true as claimed, and so (5.15), and the lemma, follow from

$$[(J-1)_{q-t-s-2}]^2 \geq (J+(n-t-1))_{q-t-s-2}(J-(n-t-1))_{q-t-s-2} . \tag{5.18}$$

Let m be the midpoint of the left side factors in (5.18):

$$\begin{aligned} m &= \frac{(J-1)+[J-1-(q-t-s-2)+1]}{2} \\ &= J - \frac{q-t-s-1}{2} = J - \frac{\lambda}{2}, \end{aligned}$$

say. Let m_1 and m_2 be the midpoints of the two sets of factors on the right side of (5.18):

$$\begin{aligned} m_1 &= J+(n-t)-\lambda/2 \\ m_2 &= J-(n-t)+2-\lambda/2 . \end{aligned}$$

We may rewrite (5.18) as

$$\prod_{h=-w}^{+w} (m+h)(m-h) \geq \prod_{h=-w}^{+w} (m_1+h)(m_2-h) , \tag{5.19}$$

where h runs over either a set of whole numbers or half integers, depending upon whether m is an integer or not. If we combine each factor in (5.19) corresponding to an index h with the factor corresponding to $-h$, we find that it will suffice to prove

$$\begin{aligned} (m^2-h^2)^2 &\geq (m_1^2-h^2)(m_2^2-h^2) \\ &= (m_1m_2-h^2)^2 - h^2(m_1-m_2)^2 . \end{aligned}$$

This is certainly true if

$$m^2 - h^2 \geq m_1m_2 - h^2 . \tag{5.20}$$

This last is also the required inequality in the event that the products on each side of (5.19) contain an index $h=0$, in which case there is no different term corresponding to $-h$ to pair it with. The inequality (5.20) is equivalent to

$$(J-\lambda/2)^2 \geq [J-\lambda/2+(n-t)][J-\lambda/2-(n-t)+2] ,$$

that is,

$$(n-t)^2 \geq 2[J - \lambda/2 + (n-t)] .$$

Since $J = \binom{n-t}{2}$, this calls for

$$\begin{aligned}(n-t)^2 &\geq (n-t)(n-t-1) - \lambda + 2(n-t)\\ &= (n-t)(n-t+1) - \lambda ,\end{aligned}$$

that is, $\lambda \geq n-t$. This is true since $\lambda = (n-t) + (k-s-1)$. This concludes the proof of the lemma. ∎

Lemma 5.4. *Uniformly for*

$$1 \leq t \leq n-1 , \ -1 \leq s \leq k/2 , \textit{ and } 0 < k \leq n^{4/3} ,$$

$$\Psi_{s,t} e^{\Psi_{s,t}} \leq \left(\frac{2xy}{n}\right)^{s+1} \left\{\frac{x+1}{2e} \sqrt{1-y^2}\right\}^t \exp\left[O(1) + O\left(\frac{stx}{n^2}\right)\right]. \tag{5.21}$$

For any given constant $C_1 < 1$, there is a positive constant ϵ_0 such that

$$\Psi_{s,t} e^{\Psi_{s,t}} = O\left(\left(\frac{2xy}{n}\right)^{s+1} \left\{\frac{1}{e}\left(1 - \frac{C_1 k}{n}\right)\right\}^t\right) \tag{5.22}$$

uniformly for

$$1 \leq t \leq n-1 , \ -1 \leq s \leq k/2 , \textit{ and } 0 < k \leq \epsilon_0 n .$$

Proof. We have

$$\Psi_{s,t} = \Psi_{s,1} \frac{\Psi_{s,2}}{\Psi_{s,1}} \cdots \frac{\Psi_{s,t}}{\Psi_{s,t-1}} \leq \Psi_{s,1}\left(\frac{\Psi_{s,2}}{\Psi_{s,1}}\right)^{t-1} , \tag{5.23}$$

where the inequality has been obtained by an application of Lemma 5.3. Using Lemma 5.2, the facts that $g_1(x) = O(x^2)$, $x^3/n = O(1)$, and $xq/n^2 = O(1)$, we have

$$\Psi_{s,1} = 2x \exp[-2x + O(1)] \frac{(q-1)_{s+1}}{(N-n-q+s+3)_{s+1}} . \tag{5.24}$$

Using Lemma 5.2 again, the big-oh observations we just made, and the fact that $g_i < 0$ for $x > 1$, we have

$$\frac{\Psi_{s,2}}{\Psi_{s,1}} \leq 2x \exp\left[-2x + O\left(\frac{1}{n}\right)\right] \frac{(q-2)_{s+1}}{(q-1)_{s+1}} \frac{(N-n-q+s+3)_{s+1}}{(N-2n-q+s+6)_{s+1}} . \tag{5.25}$$

Easily,

$$\frac{(q-2)_{s+1}}{(q-1)_{s+1}} = 1 - \frac{s+1}{q-1} \tag{5.26}$$

and

$$\begin{aligned}
&\frac{(N-n-q+s+3)_{s+1}}{(N-2n-q+s+6)_{s+1}}\\
&\quad=\prod_{j=0}^{s}\left(\frac{1-(2x+2)/(n-1)+2(s+3-j)/n(n-1)}{1-[2+(2x+2)]/(n-1)+[6+2(s+3-j)]/n(n-1)}\right)\\
&\quad=\prod_{j=0}^{s}\left(1+\frac{2/(n-1)-6/n(n-1)}{1-(2x+4)/(n-1)+[6+2(s+3-j)]/n(n-1)}\right)\\
&\quad\le\prod_{j=0}^{s}\left(1+\frac{2/n}{1-(2x+4)/(n-1)}\right)\\
&\quad=\left[1+\frac{2}{n}+O\left(\frac{x}{n^2}\right)\right]^{s+1}\\
&\quad=\exp\left\{(s+1)\left[\frac{2}{n}+O\left(\frac{x}{n^2}\right)\right]\right\}. \qquad (5.27)
\end{aligned}$$

Combining (5.25)–(5.27),

$$\begin{aligned}
\frac{\Psi_{s,2}}{\Psi_{s,1}}&\le 2x\exp\left(-2x+\frac{2s}{n}\right)\left(1-\frac{s+1}{q-1}\right)\\
&\quad\times\exp\left[O\left(\frac{sx}{n^2}\right)+O\left(\frac{1}{n}\right)\right]. \qquad (5.28)
\end{aligned}$$

Next, we have

$$\begin{aligned}
\frac{(q-1)_{s+1}}{(N-n-q+s+3)_{s+1}}&\le\left(\frac{q}{N}\right)^{s+1}\prod_{j=0}^{s}\frac{1-(j+1)/q}{1-2(x+1)/(n-1)}\\
&=\left(\frac{2x}{n}\right)^{s+1}\prod_{j=0}^{s}\left(\frac{n-(j+1)/x}{n-2x-3}\right). \qquad (5.29)
\end{aligned}$$

Now

$$\begin{aligned}
\prod_{j=0}^{s}\left(\frac{n-(j+1)/x}{n-2x-3}\right)&\le\prod_{j=0}^{s}\exp\left[-\frac{j+1}{xn}+\frac{2x+3}{n}+O\left(\frac{x^2}{n^2}\right)\right]\\
&\le\exp\left[-\frac{s^2}{2xn}+\frac{5sx}{n}+O(1)\right]\le O(1), \qquad (5.30)
\end{aligned}$$

where the rightmost inequality follows from the observation that

$$-\frac{s^2}{2xn}+\frac{5sx}{n}=\frac{s}{2xn}(-s+10x^2)$$

is negative when $s>10x^2$ and, since $x\le n^{1/3}$, is bounded otherwise. Combining (5.23), (5.24), (5.28)–(5.30), we find

$$\Psi_{s,t} \le \left\{2x \exp\left(-2x + \frac{2s}{n}\right)\left(1 - \frac{s+1}{q-1}\right) \exp\left[O\left(\frac{sx}{n^2}\right) + O\left(\frac{1}{n}\right)\right]\right\}^{t-1}$$
$$\times 2x \exp(-2x + O(1))\left(\frac{2x}{n}\right)^{s+1}$$
$$= \left(\frac{2x}{n}\right)^{s+1}\left[2x \exp\left(-2x + \frac{2s}{n}\right)\left(1 - \frac{s+1}{q-1}\right)\right]^t$$
$$\times \exp\left[O(1) + O\left(\frac{stx}{n^2}\right)\right]. \tag{5.31}$$

Next, starting with

$$\frac{q-t-s-1}{n-t} = \frac{q}{n} + \left(\frac{t(x-1)}{n-t} - \frac{s+1}{n-t}\right),$$

we use Taylor's theorem and the fact that the second derivative φ'' is negative for $x > 1$, as shown by (2.6), to find

$$\varphi\left(\frac{q-t-s-1}{n-t}\right) \le \varphi(x) + \varphi'(x)\left(\frac{t(x-1)}{n-t} - \frac{s+1}{n-t}\right).$$

By (3.2) and (3.13), we then find

$$\Phi_{s,t} \le -t\varphi(x) + t(x-1)\varphi'(x) - (s+1)\varphi'(x) + O(1).$$

Combining this with (5.31), recalling (2.5) and the definition (1.13) of β, and using $e^{2t/n} = O(1)$, we obtain

$$\Psi_{s,t}e^{\Phi_{s,t}} \le \left(\frac{2xy}{n}\right)^{s+1}\left[\left(1 - \frac{s+1}{q-1}\right)\beta e^{2s/n}\right]^t \exp\left[O(1) + O\left(\frac{stx}{n^2}\right)\right]. \tag{5.32}$$

For $q \ge 4$ and $s \le k/2$,

$$\frac{\partial \log\{[1-(s+1)/(q-1)]\, e^{2s/n}\}}{\partial s} = \frac{2}{n} - \frac{1}{q-s-2} \ge 0$$

because $q - s - 2 \ge n + k/2 - 2 \ge n/2 + (q-4)/2 \ge n/2$. Thus $(1-(s+1)/(q-1))\, e^{2s/n}$ is bounded above by its value at $s = k/2$ which is

$$\left(1 - \frac{k+2}{2q-2}\right) e^{k/n} \le \left(1 - \frac{k}{2xn}\right) e^{k/n} = \frac{x+1}{2x}\, e^{k/n}. \tag{5.33}$$

We now prove that

$$\beta = x \exp(-1 - k/n)\sqrt{1-y^2}. \tag{5.34}$$

From (2.2), (2.10), and then (1.2),

$$\beta = Te^{-T} = x(1-y)\,e^{-x+xy} = x(1-y)\,e^{-x}\sqrt{(1+y)/(1-y)}\,. \tag{5.35}$$

Replacing x by $1+k/n$ proves (5.34). Combining (5.32)–(5.34), we obtain (5.21).

We now suppose that $k \le \epsilon_0 n$ and turn our attention to (5.22). Then stx/n^2 is bounded and so, to prove (5.22), it suffices to show that

$$\frac{x+1}{2}\sqrt{1-y^2} \le 1 - \frac{C_1 k}{n}\,.$$

Since $(x+1)/2 = 1 + k/2n$, this follows from (3.1) for ϵ_0 sufficiently small. ∎

6. PROOF OF (4.6a)

We now assign to ϵ_0 and C_1 the values which they will keep for the rest of the paper. First, when x is bounded, we have, by (2.11), (2.10), and (2.2),

$$T = x(1-y) = 1 - \sqrt{3k/n} + k/n + O(k^{3/2}/n^{3/2})\,,$$

$$\beta = Te^{-T} = e^{-1}[1 - 3k/2n + O(k^{3/2}/n^{3/2})]$$

and

$$y\sqrt{n/k} = \sqrt{3} + O(k/n)\,,$$

and so we may fix ϵ_0 so small and C_1 sufficiently close to 1 that the following four conditions are satisfied:

$$\frac{1}{2e} \le \beta \le \frac{1}{e}\left(1 - \frac{C_1 k}{n}\right) \text{ for } k \le \epsilon_0 n\,, \tag{6.1}$$

$$\text{conclusion (5.2) of Lemma 5.2 holds for } \epsilon = 1/9\,, \tag{6.2}$$

$$\text{conclusion (5.22) of Lemma 5.4 holds}\,, \tag{6.3}$$

$$\frac{3(1+k/n)y}{2eC_1^{3/2}}\left(\frac{n}{k}\right)^{1/2} < 1 - \Delta \text{ for } k \le \epsilon_0 n \text{ and some } \Delta > 0\,. \tag{6.4}$$

Note that (6.4) is true for ϵ_0 close to 0 and $C_1 < 1$ close to 1 since $3\sqrt{3}/2e < 1$.

Assume that $C \ge 1$ and that the upper bound $|b(\nu, \kappa)| \le C\kappa^{1/16}/\nu^{9/50}$ holds for $(\nu, \kappa) < (n, k)$ and $\kappa \ge \nu^{1/5}$.

By Taylor's theorem

$$\varphi\left(\frac{q-1}{n}\right) = \varphi(x) - \frac{1}{n}\varphi'(x) + \frac{1}{2n^2}\varphi''(x) - \frac{1}{6n^3}\varphi'''(\xi_1)$$

for some ξ_1 between $(q-1)/n$ and q/n. Similarly,

$$a\left(\frac{q-1}{n}\right) = a(x) - \frac{1}{n}a'(x) + \frac{1}{2n^2}a''(\xi_2)\,.$$

Thus, recalling the definition (4.1a) of $\Phi_{s,t}$,

$$\Phi_{0,0} = -\varphi'(x) + \frac{1}{n}\left(\frac{1}{2}\,\varphi''(x) - a'(x)\right) + \frac{1}{n^2}\,[O(\varphi'''(\xi_1)) + O(a''(\xi_2))]\,, \tag{6.5}$$

where $x - 1/n < \xi_i < x$.

When $n^{2/9} \le k \le \epsilon_0 n$, the sum of the big-ohs in (6.5) is $O(n^2/k^2)$ by (3.6), (3.4), and the fact that $n^{1/2} \ge \epsilon_0^{-1/2} k^{1/2}$. When $k \ge \epsilon_0 n$, the sum of the big-ohs is $O(x^2(1-y))$ by (3.11) and (3.14). By (3.7) and the trivial $e^{-2x} = O(x^{-4})$, we have

$$\frac{1}{n^2}\,O(x^2(1-y)) = O(1/q^2) = O(1/k^2)\,.$$

Thus

$$A = e^{-\varphi'(x)} \exp\{(1/n)[\tfrac{1}{2}\varphi''(x) - a'(x)] + O(1/k^2)\}[1 + b(n, k-1)]\,. \tag{6.6}$$

By an argument like that in the last paragraph, but using (3.5), (3.3), (3.10), and (3.13), we see that the quantity enclosed in { } in (6.6) is $O(1/k)$. If we bound $b(n, k-1)$ by

$$\frac{C(k-1)^{1/16}}{n^{9/50}} = \frac{Ck^{1/16}}{n^{9/50}}\left[1 - \frac{1}{16k} + O\left(\frac{1}{k^2}\right)\right],$$

rewrite the exponential as

$$\{1 + (1/n)[\tfrac{1}{2}\varphi''(x) - a'(x)] + O(1/k^2)\}\,,$$

expand, bound $e^{-\varphi'(x)}$ by $O(k^{1/2}/n^{1/2})$ [as given by (2.5) and (2.11)], and rearrange, the result is (4.6a).

7. PROOF OF (4.6b)–(4.6d) FOR $x \ge 6 \log n$

We first bound $S_{-1} + S_0 + S$. Note that

$$\sum_{s=-1}^{k} c(t, t+s)c(n-t, q-t-s-1) \le \binom{N - t(n-t)}{q-1}$$

because the left side counts certain $(n, q-1)$ graphs with a particular set of $t(n-t)$ edges forbidden. Thus,

$$\sum_{t=1}^{n-1} \sum_{s=-1}^{k} \binom{n}{t} t(n-t)c(t, t+s)c(n-t, q-t-s-1)$$
$$= O(n^2)\binom{N}{q-1} \sum_{t=1}^{[n/2]} \binom{n}{t} \frac{\binom{N-t(n-t)}{q-1}}{\binom{N}{q-1}}$$

$$= O(n^2)\binom{N}{q-1}\sum_{t=1}^{[n/2]} n^t\left(\frac{N-t(n-t)}{N}\right)^{q-1}$$

$$= O(n^2)\binom{N}{q-1}\sum_{t=1}^{[n/2]} n^t\left(1-\frac{t(n/2)}{n^2/2}\right)^{q-1}$$

$$= O(n^2)\binom{N}{q-1}\sum_{t=1}^{\infty} n^t \exp[-t(q-1)/n]$$

$$= O(n^3 e^{-(q-1)/n})\binom{N}{q-1}.$$

Combining this with (3.19), we have

$$S_{-1}+S_0+S = O(n^3 e^{-x}/q)\binom{N}{q-1}\Big/\binom{N}{q} = O(n^{-4}), \tag{7.1}$$

for $x \geq 6 \log n$.

We now bound the various other terms appearing in (4.6b)–(4.6d). To begin with, note that for $x \geq 6 \log n$, using (5.35) and (3.7),

$$\begin{aligned} O(x^i\beta) &= O(x^{i+1}e^{-x}(1-y^2)^{1/2}) = O(x^{i+1}e^{-2x}) \\ &= O(e^{-x}) = O(n^{-6}), \end{aligned} \tag{7.2}$$

where the constants implied by the big-ohs depend on i. Since β is well away from $1/e$, we may bound T, U, and their derivatives by their first terms. Thus, by (7.2), (T/x) and yU/n are both $O(n^{-6})$. Recalling the definitions (1.16) of $g_i(x)$ and (4.5) of $Q(x)$, it also follows that $Q(x) = O(n^{-6})$. Combining these results with (7.1) and noting that $k^{3/2}n^{1/2} \leq n^3 n^{1/2}$, we obtain (4.6b)–(4.6d).

In view of this result and the hypothesized lower bound on k in the statement of Lemma 4.1, *we will assume that* $n^{2/9} \leq k \leq 6n \log n$ *for the remainder of the paper.*

8. PROOF OF (4.6b) FOR $x \leq 6 \log n$

In addition to ϵ_0 and C_1 having the values assigned to them at the start of Section 6, we also fix a function H for the proof of (4.6b) and (4.6c) as follows:

$$H = H(n, q) = \begin{cases} C_3 n/k^{1/2} & \text{for } n^{2/9} \leq k \leq \epsilon_0 n, \\ n^{1/8} & \text{for } \epsilon_0 n < k \leq 6n \log n, \end{cases} \tag{8.1}$$

where $C_3 > 0$ is sufficiently small that

$$\begin{aligned} 1 &+ \frac{t}{n}\left[g_1(x) + (x-1)a'(x) + \tfrac{1}{2}(x-1)^2\varphi''(x)\right] \\ &+ \frac{t(t-1)}{n}\left[g_2(x) + \tfrac{1}{2}(x-1)^2\varphi''(x)\right] > 0 \end{aligned} \tag{8.2}$$

for $1 \le t \le H$. [Recall the definition (1.16) of g_i.] For $k \le \epsilon_0 n$, this choice of C_3 is possible by (3.3), (3.5), and the fact that $g_1(x)$ and $g_2(x)$ both contain a factor of $x-1=k/n$. For $k > \epsilon_0 n$, (8.2) holds for n sufficiently large by $g_i(x)=O(x^2)$, (3.8), (3.10), and (3.13). We define

$$C_4 = \tfrac{1}{2}\min(C_1C_3, \eta), \tag{8.3}$$

where η is defined by $e^{-1-\eta} = \lambda_1(1+\epsilon_0)$ and $\lambda_1(x) = [(x+1)/2e]\sqrt{1-y^2}$. (We will see shortly that $\eta > 0$.)

Before proving (4.6b), we collect some preliminary estimates that will be useful here and in the proof of (4.6c).

Lemma 8.1. *With $T(u)$ the exponential generating function for labeled and rooted trees, as defined in* (1.14), *and $U(u)$ the same unicyclic graphs, as defined in* (1.15), *we have the following bounds, uniformly for $n^{2/9} \le k \le \epsilon_0 n$:*

$$T((1-\delta)/e) \le 1-\sqrt{\delta}, \quad \text{for } 0<\delta<1, \tag{8.4}$$

$$T^{(i)} = O((n/k)^{i-1/2}), \quad \text{for } 1 \le i \le 4, \tag{8.5}$$

$$U^{(i)} = O((n/k)^{i+1}), \quad \text{for } 0 \le i \le 2. \tag{8.6}$$

The value of η, defined above is positive. We have the following bounds for the tails of the sums defining $T^{(i)}$ and $U^{(i)}$, uniformly for $n^{2/9} \le k \le 6n\log n$ and $0 \le i \le 4$:

$$\sum_{t>H} \frac{tc(t,t-1)}{t!}\, t^i\lambda^t = O(\exp(-C_4 n^{1/9})) \tag{8.7}$$

and

$$\sum_{t>H} \frac{tc(t,t)}{t!}\, t^i\lambda^2 = O(\exp(-C_4 n^{1/9})), \tag{8.8}$$

where λ is β or either of the quantities in { } *in* (5.21) *and* (5.22).

Proof. For the first part of this proof only, we suspend the convention regarding T set forth just after (2.3), and we let T be instead only a variable, while $T(u)$ still denotes the usual function. Note that $(d/dT)(Te^{-T})$ is $(1-T)e^{-T}$, so that Te^{-T} is an increasing function of T for $0<T<1$. From (2.2), $u = T(u)\,e^{-T(u)}$, so that it suffices to show that $Te^{-T} > (1/e)(1-\delta)$ when T is set to $1-\sqrt{\delta}$. But for $T = 1-\sqrt{\delta}$, we see easily that $e^{-T} > (1/e)(1+\sqrt{\delta})$, and then

$$Te^{-T} > (1/e)(1-\sqrt{\delta})(1+\sqrt{\delta}) = (1/e)(1-\delta),$$

as was to be shown. This proves (8.4), and now we restore the usual convention regarding the notation T. By using (2.3) for $uT'(u)$, and successive differentiation, one finds that $\beta^i T^{(i)}$ is a polynomial in T divided by $(1-T)^{2i-1}$. Hence, (8.5) follows from the bound (6.1) for β and (8.4) with $\delta = C_1k/n$.

The three bounds for $U^{(i)}$ given in (8.6) are obtained similarly from (2.4). This concludes the proof of (8.6).

Finally we consider the tail sums (8.7) and (8.8). Let λ_1 and λ_2 be the values of λ from (5.21) and (5.22), respectively. By the derivation of (5.22), $\lambda_1 \le \lambda_2$. We have $[(x+1)/2x]\, e^{x-1} > 1$ for $x > 1$ because the left side is an increasing function of x. Thus, by (5.33), $\beta < \lambda_1$. Note that $(x+1)(1-y^2)^{1/2}$ is a decreasing function of x because, by (2.1),

$$2e\,\frac{d\lambda_1(x)}{dx} = \frac{(1-y^2)^{1/2}(1-x-y^2)}{1-x+xy^2} < 0\,. \tag{8.9}$$

Since (i) the sums in (8.7) and (8.8) are increasing functions of λ, (ii) $\beta < \lambda_1 \le \lambda_2$, and (iii) $\lambda_1(x)$ is a decreasing function of x, it suffices to do the following:

a. Prove (8.7) and (8.8) for λ_2 when $k \le \epsilon_0 n$.
b. Prove (8.7) and (8.8) when λ_1 is computed using $x = 1 + \epsilon_0$.

By (5.22), $\lambda_2 < e^{-1-C_1k/n}$. By (8.9), $\lambda_1(x) < \lambda_1(1) = 1/e$ and so $\lambda_1(1+\epsilon_0) = e^{-1-\eta}$ for some $\eta > 0$.

For (8.7) we note that $tc(t, t-1) = t^{t-1}$ and that $t! \sim (t/e)^t\sqrt{2\pi t}$. It follows that

$$\begin{aligned}\sum_{t\ge H} \frac{tc(t,t-1)}{t!}\, t^i\lambda^t &= O(1)\sum_{t\ge H} t^{-3/2}t^i(e\lambda)^t \\ &= O(1)\sum_{t\ge H} t^i(e\lambda)^t \\ &= O(1)\left(z\frac{d}{dz}\right)^i\left(\frac{z^H}{1-z}\right)\bigg|_{z=e\lambda} \\ &= O(H^i(e\lambda)^H(1-e\lambda)^{-i-1})\,,\end{aligned} \tag{8.10}$$

where the constant implied in the last big-oh depends on i.

We can bound $H^i(1-e\lambda)^{-i-1}$ by a polynomial in n. For case **a**,

$$(e\lambda_2)^H \le \exp(-C_1C_3k^{1/2}) \le \exp(-2C_4n^{1/9})\,.$$

Case **b** is similar and so (8.7) follows.

For (8.8), the arguments are like those in the preceding paragraph except that $tc(t,t) = O(t^{t+1/2})$ [14, Theorem 2]. This completes the proof of the lemma. ∎

We now turn to (4.6b). Assume that $C \ge 1$, $k \ge n^{2/9}$ and $|b(\nu,\kappa)| \le C\kappa^{1/16}/\nu^{9/50}$ for all $(\nu,\kappa) < (n,k)$ with $\kappa \ge \nu^{1/5}$. We start by considering those parts of the summations on t for which $t \ge H$ and showing that they can be absorbed by the big-ohs. As a result, we will be able to truncate all the sums to $t < H$.

We begin with the part of the summation for S_{-1} with $t \ge H$. Since $k \ge n^{2/9}$, certainly $k \ge (n-t)^{1/5}$, and so the above bound holds for $b(n-t,k)$ with $H < t \le n-1$. Since $C \ge 1$, we may replace $1 + b(n-t,k)$ by $O(Ck^{1/16})$. Use (5.21) with $s = -1$, and then (8.7) to find

$$\frac{1}{x} \sum_{H \le t \le n-1} \frac{tc(t, t-1)}{t!} \Psi_{-1,t} \exp(\Phi_{-1,t})[1 + b(n-t, k)]$$

$$\le O(Ck^{1/16}) \sum_{H < t \le n-1} \frac{tc(t, t-1)}{t!} \lambda^t$$

$$= O(Ck^{-3/16} n^{-1/2} \exp(-C_4 n^{1/9})) = O(C \exp(-C_4 n^{1/9})) . \qquad (8.11)$$

We may convert (4.5) to an infinite sum on t by replacing $\beta T'$ and $\beta^2 T''$ with

$$\beta^i T^{(i)} = \sum_{t=1}^{\infty} (t)_i \frac{tc(t, t-1)}{t!} \beta^t . \qquad (8.12)$$

Let $Q_H(x)$ be the result of making this replacement and letting the sum range over $H \le t < \infty$. By (1.16) and the bounds in Lemmas 3.1 and 3.2

$$Q_H(x) = O(x) \sum_{t \ge H} t \frac{tc(t, t-1)}{t!} \beta^t + O(1) \sum_{t > H} t^2 \frac{tc(t, t-1)}{t!} \beta^t . \qquad (8.13)$$

By (8.7) and (8.13), $Q_H(x) = O(k^{-3/2} n^{-1/2})$. Estimate the tails of T and $(1/xn)\beta T'$ similarly. Combining these results with (8.11), we see that it suffices to prove (4.6b) where all sums over t are truncated to run over $t < H$.

We limit our attention to $t < H$ for the rest of this section. Expressions coming from sums of $t < H$ will be indicated with a subscript L as in $S_{-1,L}$. We will use Lemma 5.2 to estimate $S_{-1,L}$.

We have

$$\frac{q-t}{n-t} = \frac{xn - xt}{n-t} + \frac{xt - t}{n-t} = x + \frac{t(x-1)}{n-t} ,$$

and so by Taylor's theorem

$$\varphi\left(\frac{q-t}{n-t}\right) = \varphi(x) + \frac{t(x-1)}{n-t} \varphi'(x) + \frac{1}{2} \left(\frac{t(x-1)}{n-t}\right)^2 \varphi''(x) + \frac{1}{6} \left(\frac{t(x-1)}{n-t}\right)^3 \varphi'''(\xi_3) ,$$

where ξ_3 is between q/n and $(q-t)/(n-t)$, so that, by (3.6),

$$\varphi'''(\xi_3) = O(n^2/k^2) .$$

Likewise

$$a\left(\frac{q-t}{n-t}\right) = a(x) + \frac{t(x-1)}{n-t} a'(x) + \frac{1}{2} \left(\frac{t(x-1)}{n-t}\right)^2 a''(\xi_4)$$

and, by (3.4) and (3.14),

$$(x-1)^2 a''(\xi_4) = O(1) .$$

Substituting $(n-t)\varphi((q-t)/(n-t))$ into the expression for $\Phi_{-1,t}$ and setting $1/(n-t) = 1/n + O(t/n^2)$, we find, using $(x-1) = k/n$, (3.3), (2.6), (3.10), (3.13), and $t < H$, that

$$\begin{aligned}\Phi_{-1,t} &= -t\varphi(x) + t(x-1)\varphi'(x) + (1/n)[\tfrac{1}{2}t^2(x-1)^2\varphi''(x) + t(x-1)a'(x)] \\ &\quad + t^3O(k/n^3) + t^2O(1/n^2) \\ &= -t\varphi(x) + t(x-1)\varphi'(x) + O(1). \end{aligned} \tag{8.14}$$

Observing that $x - 1 = k/n$ and that $g_1(x)$ and $g_2(x)$ both contain $(x-1)$ as a factor, we see that the quantity in [ ] on the right side of (5.2) is $-2xt + O(1)$. We may combine (5.2) and (8.14) and expand the exponential to obtain

$$\Psi_{-1,t}\exp(\Phi_{-1,t}) = \{2x\exp[-2x - \varphi(x) + (x-1)\varphi'(x)]\}^t[1 + \Lambda + O(\Lambda^2)], \tag{8.15}$$

where

$$\begin{aligned}\Lambda &= (t/n)[g_1(x) + (x-1)a'(x) + \tfrac{1}{2}(x-1)^2\varphi''(x)] \\ &\quad + (t(t-1)/n)[g_2(x) + \tfrac{1}{2}(x-1)^2\varphi''(x)] \\ &\quad + O(t^2/n^2) + O(t^3k/n^3) + O(t^4k^2/n^4). \end{aligned} \tag{8.16}$$

By (1.16), (3.3), (2.6), (3.10), and (3.13), Λ^2 plus the big-ohs in (8.16) is

$$O(t^3x^3k/n^3) + O(t^2/n^2) + O(t^4k^2/n^4).$$

If we multiply (8.15) by $(1/x)tc(t, t-1)t!$ and sum over $t < H$, we obtain

$$\begin{aligned}&\frac{1}{x}\sum_{t<H}\frac{tc(t,t-1)}{t!}\Psi_{-1,t}\exp(\Phi_{-1,t}) \\ &= \frac{1}{x}T_L + \frac{1}{n}Q_L(x) + O\left(\frac{\beta^3T^{(3)}x^3k}{n^3}\right) + O\left(\frac{\beta^2T^{(2)}}{n^2}\right) + O\left(\frac{\beta^4T^{(4)}k^2}{n^4}\right). \end{aligned} \tag{8.17}$$

We claim that the big-ohs in (8.17) are $O(k^{-3/2}n^{-1/2})$. If $k \le \epsilon_0 n$, use (8.5) to prove it. Suppose now that $k > \epsilon_0 n$. We easily have $T^{(i)} = O(1)$. The desired bounds now follow from (5.34). It follows by this and the induction hypothesis in Lemma 4.1 that, to prove (4.6b), it suffices to show that

$$\left|\sum_{t<H}\frac{tc(t,t-1)}{t!}\Psi_{-1,t}\exp(\Phi_{-1,t})\frac{Ck^{1/16}}{(n-t)^{9/50}}\right| \tag{8.18}$$

is equal to the right side of (4.6b). Since all the terms in the sum are positive, the absolute values are not needed. We write

$$\frac{1}{(n-t)^{9/50}} = \frac{1}{n^{9/50}}\left[1 + \frac{9t}{50n} + O\left(\frac{t^2}{n^2}\right)\right]$$

and proceed as in the previous paragraph. Corresponding to (8.17), we have

$$(8.18) = \frac{Ck^{1/16}}{n^{9/50}}\left[\frac{1}{x}T_L + \frac{1}{n}Q(x) + \frac{9}{50}\frac{1}{xn}\beta T' + O\left(\frac{\beta^3 T^{(3)}x^3 k}{n^3}\right) + O\left(\frac{\beta^2 T^{(2)}}{n^2}\right) + O\left(\frac{\beta^4 T^{(4)}k^2}{n^4}\right)\right].$$

By choosing $C_2 < C_4$ sufficiently small that $1/C_2$ accounts for all constants implied by big-ohs in the preceding, we conclude that (4.6b) holds.

9. PROOF OF (4.6b) FOR $x \le 6 \log n$

The arguments in this section are very similar to those in the previous section. Since $k \ge n^{2/9}$, $k - 1 \ge (n-t)^{1/5}$ and so the induction hypothesis in Lemma 4.1 can be applied to $b(n-t, k-1)$. Using (5.21) with $s = 0$ and then (8.8), we find, in place of (8.12),

$$\begin{aligned}\frac{1}{x}\sum_{t\ge H}\frac{tc(t,t)}{t!}\Psi_{0,t}\exp(\Phi_{0,t})[1 + b(n-t, k-1)] &\\ \le O\left(\frac{Cyk^{1/16}}{n}\right)\sum_{t\ge H}\frac{tc(t,t)}{t!}e^{-t}\lambda^t &\\ = O\left(Ck^{9/16}n^{-3/2}\left(\frac{n}{k}\right)\exp(-C_2 n^{1/9})\right) &\\ = O(C\exp(-C_2 N^{1/9}))\,. &\qquad (9.1)\end{aligned}$$

As in the previous section, we see that it suffices to prove (4.6c) with all sums truncated to $t < H$.

We limit our attention to $t < H$ for the rest of this section. From $(q-t-1)/(n-t) = x + [t(x-1)-1]/(n-t)$ we have the two Taylor expansions

$$\varphi\left(\frac{q-t-1}{n-t}\right) = \varphi(x) + \frac{t(x-1)-1}{n-t}\varphi'(x) + \frac{1}{2}\left(\frac{t(x-1)-1}{n-t}\right)^2\varphi''(\xi_5) \qquad (9.2)$$

and

$$a\left(\frac{q-t-1}{n-t}\right) = a(x) + \frac{t(x-1)-1}{n-t}a'(\xi_6)\,, \qquad (9.3)$$

and then, as in (8.14),

$$\begin{aligned}\Phi_{0,t} &= -t\varphi(x) + t(x-1)\varphi'(x) - \varphi'(x) + O(t^2k/n^2) + O(1/k) + O(t/n)\\ &= -t\varphi(x) + t(x-1)\varphi'(x) - \varphi'(x) + O(1)\,.\end{aligned}$$

Using (5.2) with the exponentiated quantity in [ ] shortened to $-2xt + O(t^2kx/n^2)$, combining with (9.3), expanding the exponential, and noting that the quantity $(q-t)/(N-B-q+1)$ appearing in (5.2) equals $(2x/n)[1+O(tx/n)]$, we have

$$\Psi_{0,t}\exp(\Phi_{0,t}) = (2xy/n)\{2x\exp[-2x-\varphi(x)+(x-1)\varphi'(x)]\}^t \times [1+O(1/k)+O(tx/n)+O(t^2kx/n^2)] \tag{9.4}$$

for $t \le H$.

Using (9.4) in place of (8.15), we obtain, in place of (8.17), $(1/x)tc(t,t)/t!$, summing over $t<H$, and using (8.6) estimates, we obtain

$$\frac{1}{x}\sum_{t<H}\frac{tc(t,t)}{t!}\Psi_{0,t}\exp(\Phi_{0,t}) = \frac{2y}{n}\left\{U_L + O\left(\frac{U}{k}\right) + O\left(\frac{\beta x U^{(1)}}{n}\right) + O\left(\frac{\beta^2 kxU^{(2)}}{n^2}\right)\right\} = \frac{2y}{n}U_L + O(k^{-3/2}n^{-1/2}). \tag{9.5}$$

Thus it suffices to show that (4.6c) is valid when the left side is replaced by

$$\left|\frac{2y}{n}\sum_{t<H}\frac{tc(t,t)}{t!}\Psi_{0,t}\exp(\Phi_{0,t})b(n-t,k-1)\right|. \tag{9.6}$$

Bounding $b(n-t,k-1)$ with $Ck^{1/16}/n^{9/50}[1+O(t/n)]$, expanding the product, and proceeding as in (9.5), we find that

$$(9.6) = \frac{2y}{n}\frac{Ck^{1/16}}{n^{9/50}}[U_L + O(k^{-3/2}n^{-1/2})],$$

which gives (4.6c).

10. PROOF OF (4.6d) FOR $x \le 6\log n$

We need the following estimate later.

Lemma 10.1. *Let $p(n)$ stand for a polynomial in n, not necessarily the same at each occurrence. When $x = O(\log n)$ and $s_0 > n^{4/5}$,*

$$\frac{1}{\binom{N}{q}}\sum_{\substack{s_0\le s\le (k-1)/2\\ 1\le t\le n-1}}\binom{n}{t}\binom{\binom{t}{2}}{t+s}\binom{\binom{n-t}{2}}{q-t-s-1} = O(p(n))2^{-s_0}. \tag{10.1}$$

Proof. It suffices to prove the lemma with q replaced by $q+1$ and $\binom{N}{q+1}$ replaced by $\binom{N}{q}$. Let $t=(ns/k)(1+\delta)$. Note that

$$n/2 > ns/k > ns_0/q > n^{3/4}.$$

We will replace $\binom{n}{t}$ by the exponential parts of Stirling's formula and the three binomial coefficients of the form $\binom{M}{b}$ with $M = \binom{m}{2}$ by

$$\frac{M^b}{b!}\prod_{i<b}\left(1-\frac{i}{M}\right)=\frac{M^b}{b!}\exp\left[-\frac{b^2}{m^2}+O\left(\frac{b^3}{m^4}\right)\right]$$
$$=\frac{m^{2b}}{2^b b!}\exp\left[-\frac{b}{m}-\frac{b^2}{m^2}+O\left(\frac{b^3}{m^4}\right)\right].$$

For the three binomial coefficients, we have $b^3/m^4=o(1)$, $b/m=x+o(1)$, and $b^2/m^2=x^2+o(1)$ when δ is small. Putting this all together, using Stirling's formula for $b!$ and doing some rearranging, we obtain

$$\binom{n}{t}\binom{\binom{t}{2}}{t+s}\binom{\binom{n-t}{2}}{n-t+(k-s)}=O(p(n))\exp(-2x-2x^2)$$
$$\times\left(\frac{k}{s}\right)^{ns/k}\left(\frac{k}{s}\right)^{ns\delta/k}(1+\delta)^{-ns(1+\delta)/k}$$
$$\times\left(\frac{k}{k-s}\right)^{n(k-s)/k}\left(\frac{k}{k-s}\right)^{-ns\delta/k}\left(1-\frac{s\delta}{k-s}\right)^{-n(k-s-s\delta)/k}$$
$$\times\left(\frac{en^2s}{2kq}\right)^{sq/k+ns\delta/k}\left(\frac{(1+\delta)^2}{1+n\delta/q}\right)^{sq/k+ns\delta/k}$$
$$\times\left(\frac{en^2(k-s)}{2kq}\right)^{(k-s)q/k-ns\delta/k}\left\{\frac{[1-s\delta/(k-s)]^2}{1-ns\delta/q(k-s)}\right\}^{(k-s)q/k-ns\delta/k}$$

Moving all δ's to the exponent by using the Taylor series for $\log(1+u)$ and then rearranging, we obtain

$$\binom{n}{t}\binom{\binom{t}{2}}{t+s}\binom{\binom{n-t}{2}}{n-t+(k-s)}=O(p(n))\exp(-2x-2x^2)\left(\frac{s}{k}\right)^s\left(\frac{k-s}{k}\right)^{k-s}$$
$$\times\left(\frac{en^2}{2q}\right)^q\exp\left[\frac{-3ks\delta^2}{2(k-s)}\right]$$

for small δ. Introducing the approximation for $\binom{N}{q}$ mentioned above and using the fact that $s_0\le s<k/2$, we see that

$$\frac{1}{\binom{N}{q}}\binom{n}{t}\binom{\binom{t}{2}}{t+s}\binom{\binom{n-t}{2}}{n-t+(k-s)}=O(p(n))2^{-s_0}\qquad(10.2)$$

for small enough δ. Summing on t and then on s, we see that (10.2) contributes at most $O(p(n))2^{-s_0}$. For larger values of $|\delta|$, the contribution to the sum is negligible. (One can prove this by using the ratio of the s, t term to the s, $t+1$ term to show monotonicity when $|\delta|$ is not small.) ∎

Proof of (4.6*d*) *for Small k.* Assume that $C\ge 1$, that $n^{2/9}\le k\le\epsilon_0 n$, and that $|b(\nu,\kappa)|\le C\kappa^{1/16}/\nu^{9/50}$ for all $(\nu,\kappa)<(n,k)$ with $\kappa\ge\nu^{1/5}$.

In the definition (4.3) of S, replace $c(n-t,q-t-s-1)$ by its equivalent form from (1.19). By restricting the range of s, we may omit the factor $\frac{1}{2}$. We find the following upper bound:

$$S \le \frac{1}{x} \sum_{\substack{1 \le s \le (k-1)/2 \\ 1 \le t \le n-1}} \frac{tc(t, t+s)}{t!} \Psi_{s,t} \exp(\Phi_{s,t})[1 + b(n-t, k-s-1)]\,. \tag{10.3}$$

For those values of t and s in the double summation for which $k - s - 1 \ge (n-t)^{1/5}$, we know that $|b(n-t, k-s-1)| \le C(k-s-1)^{1/16}/(n-t)^{9/50}$, which in turn is $O(Ck^{1/16})$. For those values of s and t in the double summation for which $k - s - 1 \le (n-t)^{1/5}$, we know, by Lemma 3.5, that $|b(n-t, k-s-1)| = O(1/(k-s-1)) = O(1)$. (We could say $O(1/k)$ for the last, since $s < k/2$, but this is not necessary.) Thus, since $C \ge 1$,

$$S \le O(Ck^{1/16}) \sum_{\substack{1 \le s \le (k-1)/2 \\ 1 \le t \le n-1}} \frac{tc(t, t+s)}{t!} \Psi_{s,t} \exp(\Phi_{s,t})\,. \tag{10.4}$$

Invoking (5.22) in Lemma 5.4,

$$S \le O(Ck^{1/16}) \sum_{\substack{1 \le s \le (k-1)/2 \\ 1 \le t \le n-1}} \frac{tc(t, t+s)}{t!} e^{-t}\left(1 - \frac{C_1 k}{n}\right)^t \left(\frac{2xy}{n}\right)^{s+1}.$$

According to the second inequality of [14, (2.5)], $c(t, t+s)/t!$ is at most b_s times the coefficient of u^t in $[1 - T(u)]^{-3s}$, where b_s is a sequence of numbers defined by [14, (3.1)–(3.4)], and we note that Wright's $\Theta(u)$ is equal to our $1 - T(u)$. It follows by differentiation, (2.3), and setting u equal to $(1/e)(1 - C_1 k/n)$, that

$$\sum_{t=1}^{n-1} \frac{tc(t, t+s)}{t!} e^{-t}\left(1 - \frac{C_1 k}{n}\right)^t \le 3sb_s \left.\frac{T(u)}{[1 - T(u)]^{3s+2}}\right|_{u=(1-C_1 k/n)/e}. \tag{10.5}$$

Since $1 - T((1 - C_1 k/n)/e) \ge \sqrt{C_1 k/n}$ by (8.4), we have

$$S \le O(Ck^{1/16}) \sum_{s=1}^{(k-1)/2} sb_s \left(\frac{n}{C_1 k}\right)^{(3s+2)/2} \left(\frac{2xy}{n}\right)^{s+1}.$$

It was shown in [14] that $sb_s = (3/2)^s s!\, O(1)$. Thus

$$S \le O(Ck^{1/16}) \frac{2xy}{k} \sum_{s=1}^{(k-1)/2} s! \left(\frac{3n^{3/2} 2xy}{2(C_1 k)^{3/2} n}\right)^s. \tag{10.6}$$

If we use $(s/e)^s O(\sqrt{s})$ for $s!$ in the previous summation, we see that the sth term will be $O(\sqrt{s})$ times the sth power of the quantity

$$3sn^{1/2} xy/e(C_1 k)^{3/2}\,.$$

If we bound s/k by $1/2$, then the latter is seen to be bounded away from 1, precisely by the condition (6.4). Hence the summation in (10.6) is O(the first term), which is $O(1/k)$, and this gives us the desired relation (4.6d) in the case of small k. ∎

Proof of (4.6d) *for Large k.* Assume now that $C \geq 1$, that $\epsilon_0 n \leq k \leq 6n \log n$, and that $|b(\nu, \kappa)| \leq C\kappa^{1/16}/\nu^{9/50}$ for all $(\kappa, \nu) < (n, k)$ with $\kappa \geq \nu^{1/5}$. In the definition (4.3) of S, we eliminate the factor of 1/2 as in (10.3) and partition the sum into low s and high s, writing

$$S \leq \sum_{1 \leq s < s_0} \cdots + \sum_{s_0 \leq s \leq (k-1)/2} \cdots = S_{\text{low}} + S_{\text{high}} ,$$

where

$$s_0 = \lceil C_5 n/x \rceil , \tag{10.7}$$

C_5 being a sufficiently small constant whose exact value will be specified shortly.

We will bound S_{low} in very much the same way that the entire sum S was bounded in the first part of this section. Equation (10.4) remains valid provided S is replaced by S_{low} and the range of s in the summation is restricted to $1 \leq s \leq s_0$. Since $sx = o(n^2)$ in this range, we can rewrite (5.21) as

$$\Psi_{s,t} \exp^{\Phi_{s,t}} = O(1)\left(\frac{2xy}{n}\right)^{s+1}\left\{[1 + o(1)]\,\frac{x+1}{2e}\,\sqrt{1-y^2}\right\}^t .$$

This gives

$$S_{\text{low}} = O(Ck^{1/16}xy/n) \sum_{\substack{1 \leq s \leq s_0 \\ 1 \leq t \leq n-1}} \frac{tc(t, t+s)}{t!}\left(\frac{2xy}{n}\right)^s \Lambda^t , \tag{10.8}$$

where $\Lambda = [1 + o(1)]\lambda_1$ and $\lambda_1 = \sqrt{1-y^2}(x+1)/2e$. Since $x \geq 1 + \epsilon_0$ and $d\lambda_1/dy < 0$ by (8.9), it follows that Λ is bounded away from $1/e$. Thus $T(\Lambda)$ is bounded away from 1. We apply Wright's bound in (10.8), i.e., we use (10.5) with $e^{-1}(1 - C_1 k/n)$ replaced by Λ, and find

$$S_{\text{low}} = O(Ck^{1/16}xyT(\Lambda)/n) \sum_{1 \leq s \leq s_0} sb_s(2xy/n)^s[1 - T(\Lambda)]^{-3s} . \tag{10.9}$$

As was done in obtaining (10.6), we bound sb_s by $O(1)s!\,(3/2)^s$. Much as before, we find that the sum in (10.9) is big-oh of the first term, provided only that for some fixed $C_6 < 1$ and all $s \leq s_0$

$$\frac{2xy}{n}[1 - T(\Lambda)]^{-3}\,\frac{3s}{2e} < C_6 . \tag{10.10}$$

As already noted, $[1 - T(\Lambda)]^{-1} = O(1)$. Thus (10.10) follows from (10.7) provided that C_5 is sufficiently small. Thus

$$S_{\text{low}} = O(Ck^{1/16}x^2y^2T(\Lambda)/n^2) = O(Ck^{1/16}x^{7/2}y^2T(\Lambda)/q^{3/2}n^{1/2}) .$$

Since $\Lambda = O(x(1-y)^{1/2})$ and $T(\Lambda) = O(\Lambda)$, it follows from (3.8) that $x^{7/2}T(\Lambda)$ is bounded. This proves that

$$S_{\text{low}} = O(Ck^{1/16}/k^{3/2}n^{1/2}) .$$

We now turn to S_{high}. By (3.29) for $c(t, t+s)$ and the induction hypothesis $|b(\nu, \kappa)| \le C\kappa^{1/16}/\nu^{9/50}$ with $\nu = n - t$ and $\kappa = k - s - 1$,

$$S_{\text{high}} = O(Ck^{1/16}n^{3/2}) \sum_{\substack{s_0 \le s \le (k-1)/2 \\ 1 \le t \le n-1}} \binom{n}{t} t(n-t) \times \left[\binom{\binom{t}{2}}{t+s} \binom{\binom{n-t}{2}}{q-t-s-1} \Big/ q\binom{N}{q} \right] t^{3/2} e^{z}, \qquad (10.11)$$

where

$$z = a\left(\frac{q-t-s-1}{n-t}\right) + t\varphi\left(\frac{t+s}{t}\right) + (n-t)\varphi\left(\frac{q-t-s-1}{n-t}\right) - n\varphi\left(\frac{q}{n}\right).$$

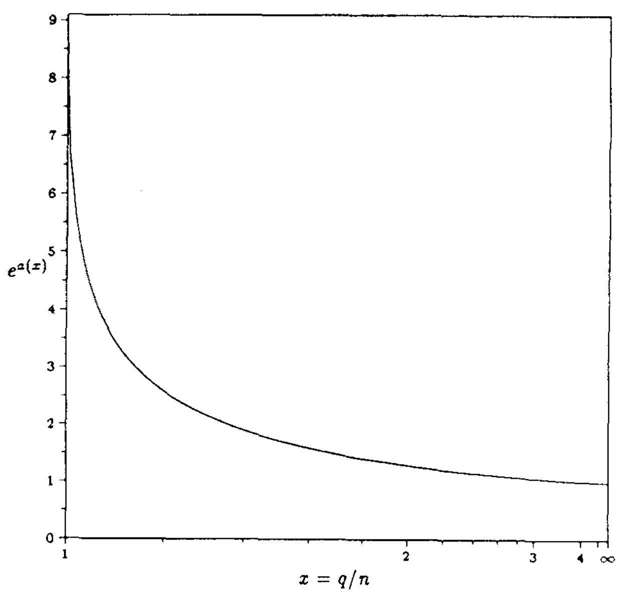
Since a is a bounded function and φ is a concave increasing function, we have

$$z \le O(1) + n\varphi\left(\frac{q-1}{n}\right) - n\varphi\left(\frac{q}{n}\right) \le O(1).$$

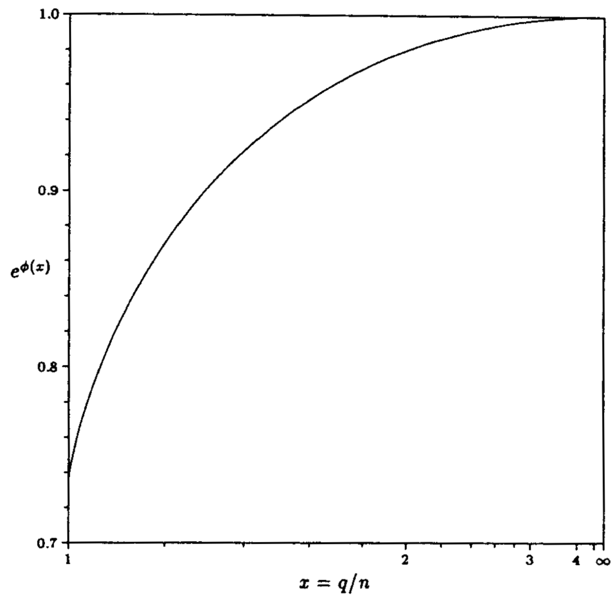
Combining this with (2.5) and (10.11), we find

$$S_{\text{high}} = O(Ck^{1/16}n^{3}) \sum_{\substack{s_0 \le s \le (k-1)/2 \\ 1 \le t \le n-1}} \binom{n}{t} t(n-t) \binom{\binom{t}{2}}{t+s} \binom{\binom{n-t}{2}}{q-t-s-1} \Big/ \binom{N}{q}.$$

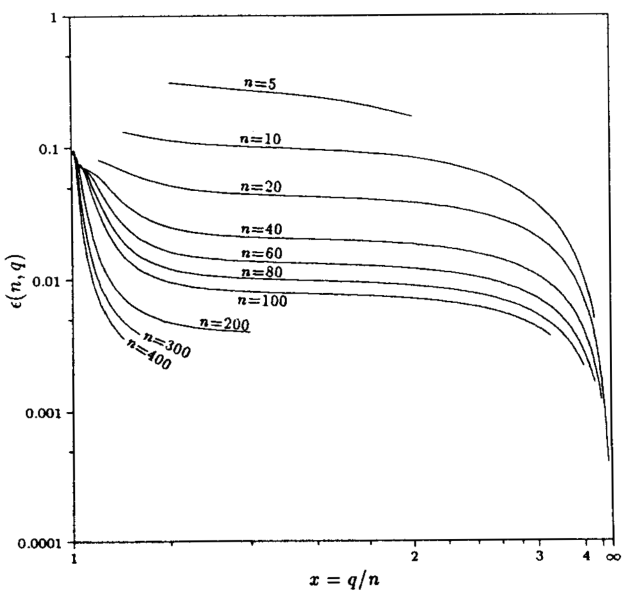
By Lemma 10.1 and (10.7), we see that $S_{\text{high}} = O(Cn^{-u})$ for all fixed u.



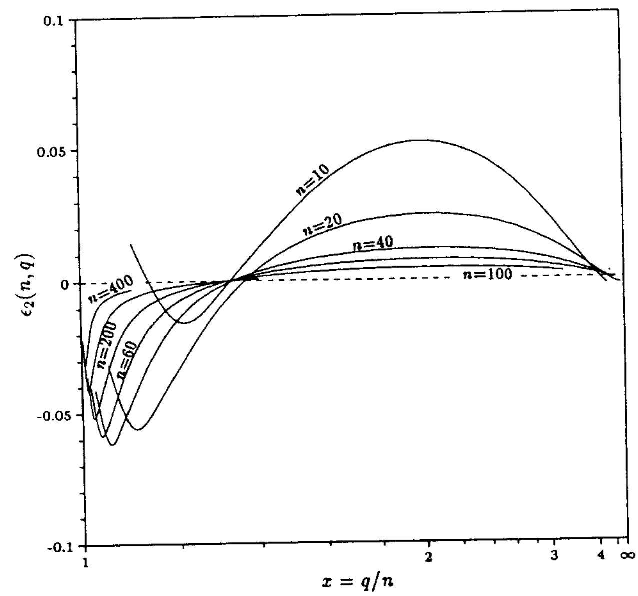
exp($a(x)$) as a function of x



$\exp(\phi(x))$ as a function of x



$\epsilon(n,q)$ as a function of x for various n



$\epsilon_2(n, q)$ as a function of x for various n

APPENDIX: WHERE SYMBOLS ARE DEFINED

$<$	after (4.5)
$a(x)$	(1.5)
A	(4.3)
B	(4.7) (6.4)
C	Lemma 4.1
C_1	(6.1–4)
C_2	Lemma 4.1
C_3	(7.1)
C_4	(8.3)
C_5	(10.7)
d_k	(1.7)
$g_i(x)$	(1.16)
H	(8.1)
k	(1.1)
K	(1.36)
N	(1.1)
n_0	Lemma 4.1
$P(x)$	(4.4)
$Q(x)$	(4.5)
S	(4.3)
S_{-1}	(4.3)
S_0	(4.3)
T	$T(\beta)$ (2.9)
w_k	(1.6)
x	(1.1)
y	(1.2) (2.5)
β	(1.13)
$\epsilon(n, q)$	Remark 2 in Section 1
ϵ_0	(5.22) (6.1–4)
$\epsilon_2(n, q)$	Remark 2 in Section 1
φ	(1.12)
e^{φ}	(2.8)
Φ	(4.1a)
Ψ	(4.1b)

□

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